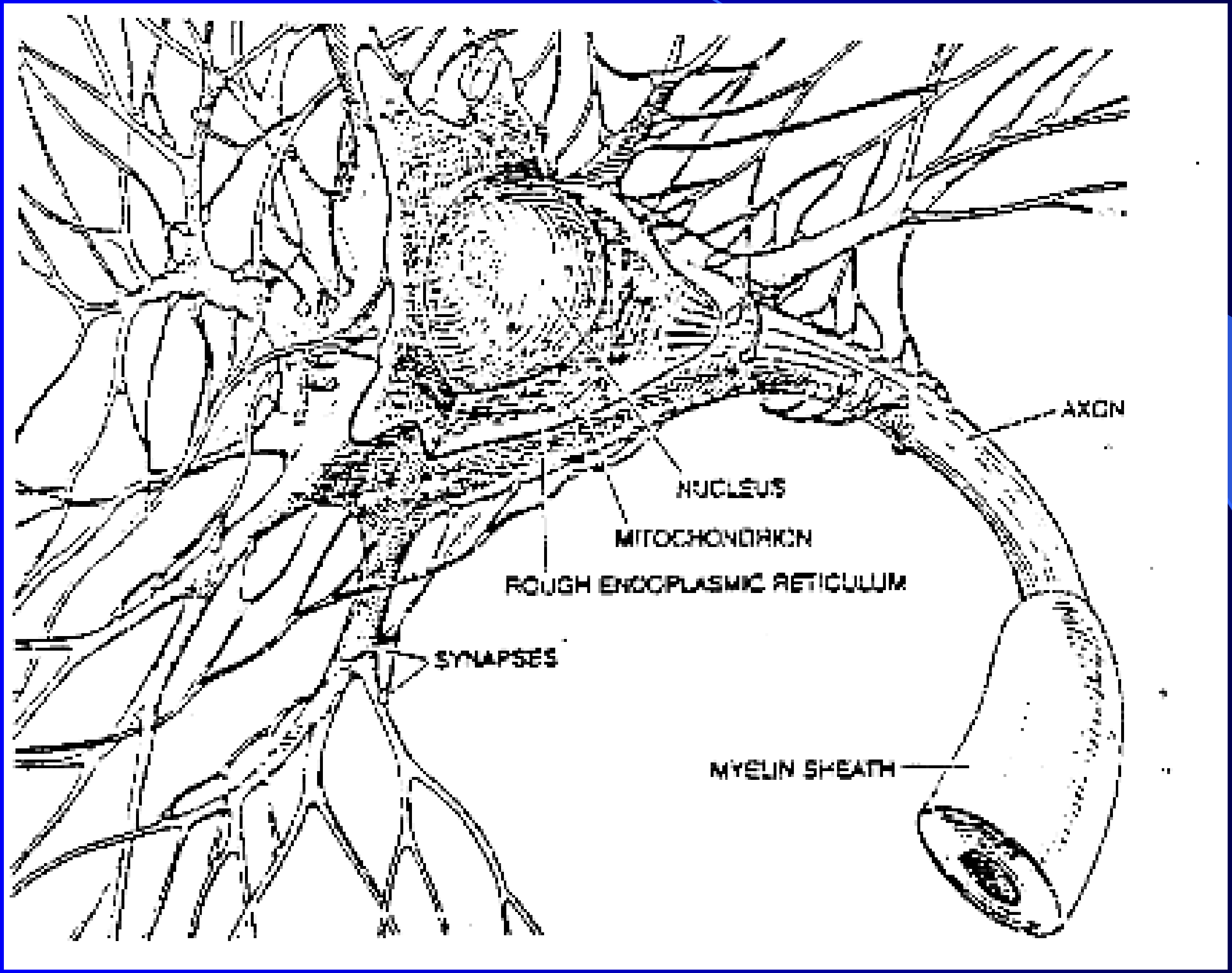
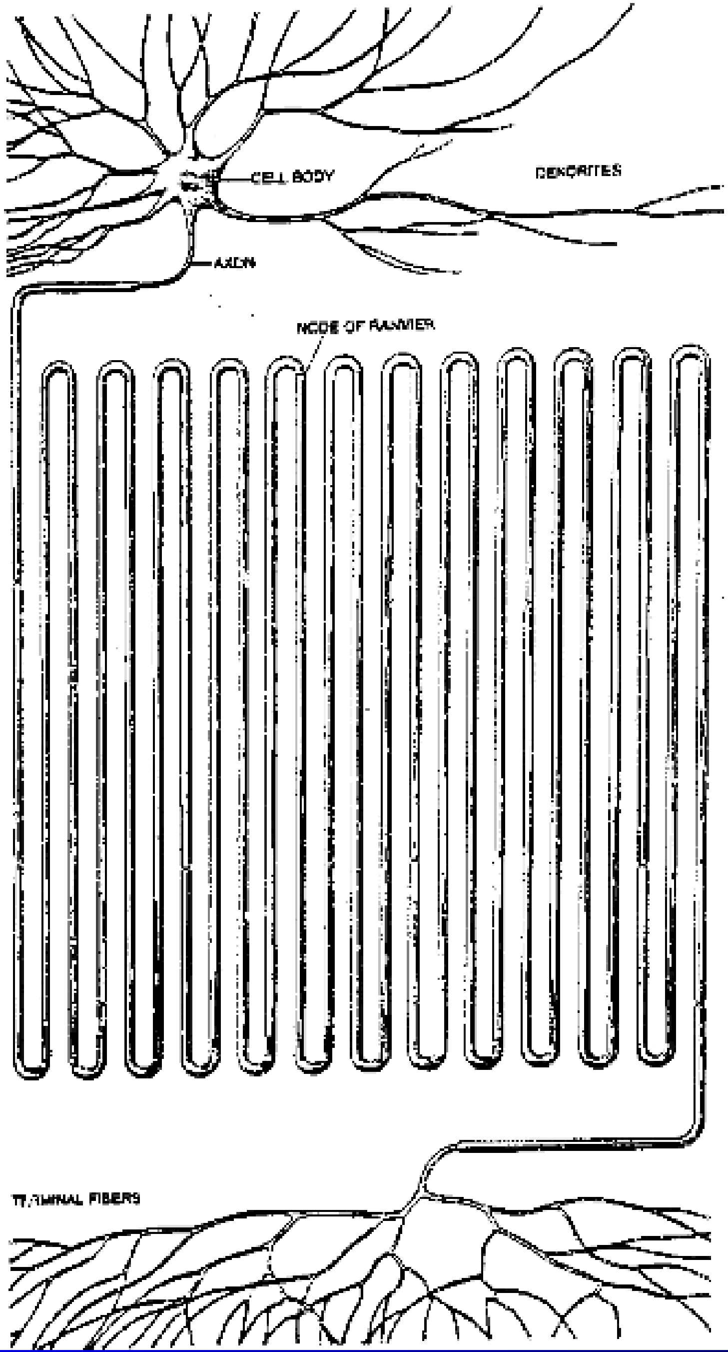
Basic Neuron



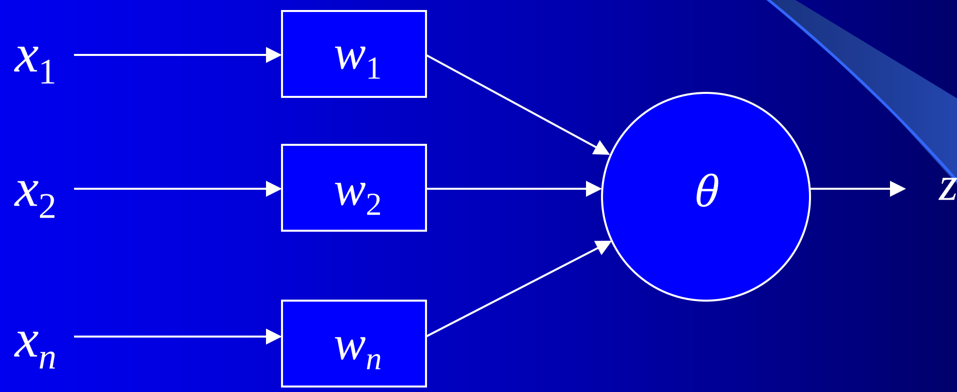
Generic Neuron and Neurites



Perceptron Learning Algorithm

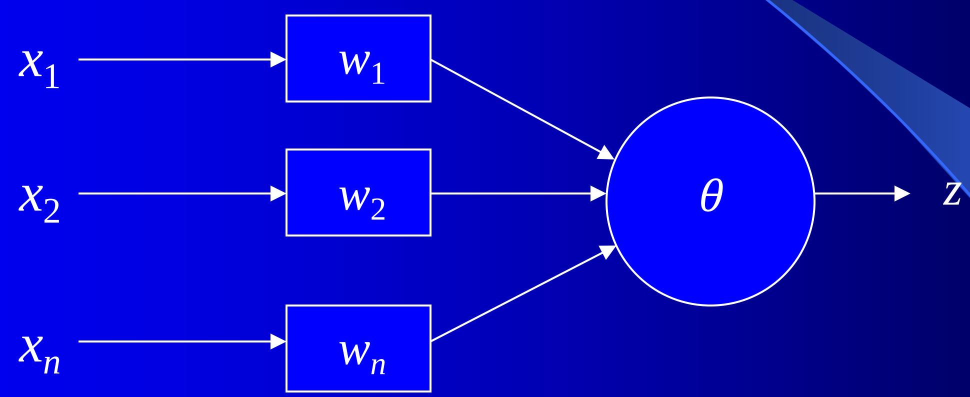
- First neural network learning model in the 1960's
  - Frank Rosenblatt
- Simple and limited (single layer model)
- Basic concepts are similar for multi-layer and deep models so this is a good learning tool
- Still used in some current applications (large business problems, where intelligibility is needed, etc.)

Perceptron Node – Threshold Logic Unit



$$z = \begin{cases} 1 & \text{if } \sum_{i=1}^{n} x_i w_i \geq \theta \\ 0 & \text{if } \sum_{i=1}^{n} x_i w_i < \theta \end{cases}$$

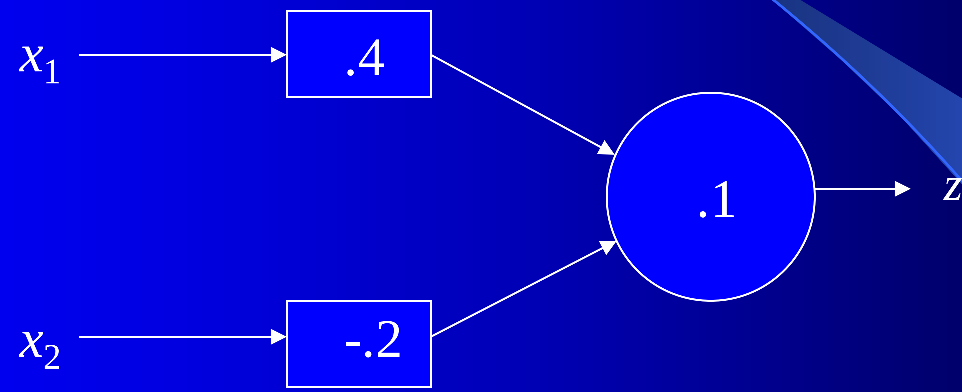
Perceptron Node – Threshold Logic Unit



- Learn weights such that an objective function is maximized.
- What objective function should we use?
- What learning algorithm should we use?

$$z = \begin{cases} 1 & \text{if } \sum_{i=1}^{n} x_i w_i \geq \theta \\ 0 & \text{if } \sum_{i=1}^{n} x_i w_i < \theta \end{cases}$$

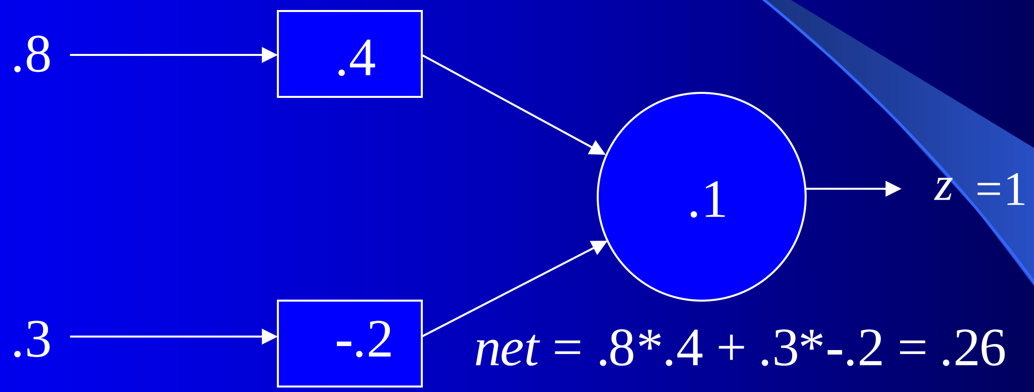
Perceptron Learning Algorithm



x_1	x_2	t
.8	.3	1
.4	.1	0

$$z = \begin{cases} 1 & \text{if } \sum_{i=1}^{n} x_i w_i \geq \theta \\ 0 & \text{if } \sum_{i=1}^{n} x_i w_i < \theta \end{cases}$$

First Training Instance

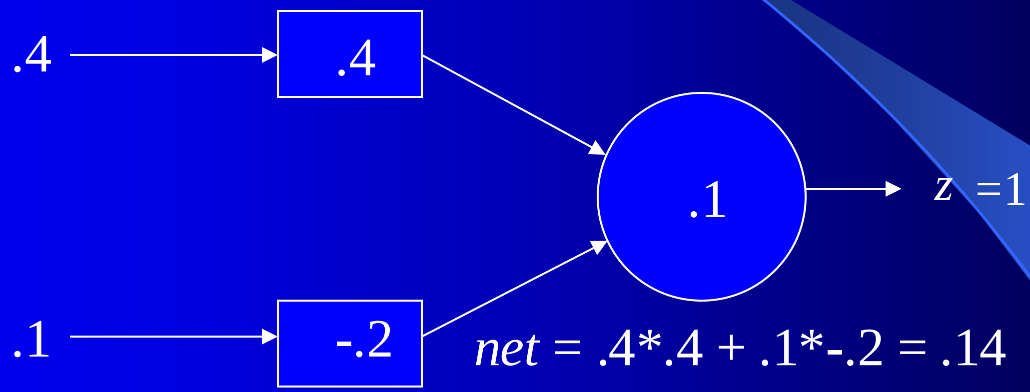


$net = .8*.4 + .3*-.2 = .26$

x_1	x_2	t
.8	.3	1
.4	.1	0

$$z = \begin{cases} 1 & \text{if } \sum_{i=1}^{n} x_i w_i \geq \theta \\ 0 & \text{if } \sum_{i=1}^{n} x_i w_i < \theta \end{cases}$$

Second Training Instance



$$net = .4*.4 + .1*-.2 = .14$$

x_1	x_2	t
.8	.3	1
.4	.1	0

$$z = \begin{cases} 1 & \text{if } \sum_{i=1}^{n} x_i w_i \geq \theta \\ 0 & \text{if } \sum_{i=1}^{n} x_i w_i < \theta \end{cases}$$

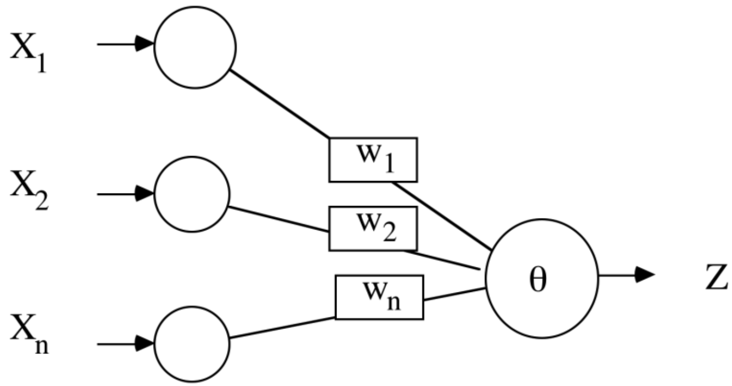
$$\Delta w_i = (t - z) * c * x_i$$

Perceptron Rule Learning

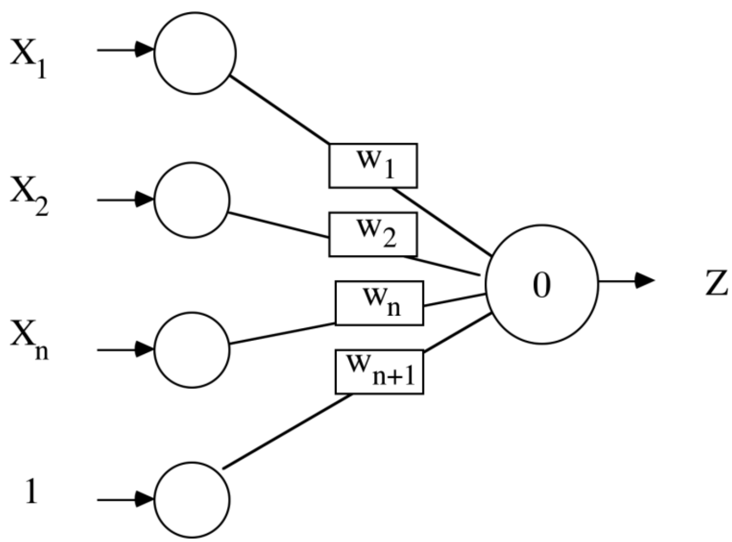
$$\Delta w_i = c(t - z)\, x_i$$

- Where w_i is the weight from input i to the perceptron node, c is the learning rate, t is the target for the current instance, z is the current output, and x_i is i^{th} input
- Least perturbation principle
  - Only change weights if there is an error
  - small c rather than changing weights sufficient to make current pattern correct
  - Scale by x_i
- Create a perceptron node with n inputs
- Iteratively apply a pattern from the training set and apply the perceptron rule
- Each iteration through the training set is an *epoch*
- Continue training until total training set error ceases to improve
- Perceptron Convergence Theorem: Guaranteed to find a solution in finite time if a solution exists

Weight Versus Threshold



Do you need to adjust Theta? Yes, in most cases



where $w_{n+1} = -\theta$

Augmented Pattern Vectors

1 0 1 -> 0

1 0 0 -> 1

Augmented Version

1 0 1 1 -> 0

1 0 0 1 -> 1

- Treat threshold like any other weight. No special case. Call it a *bias* since it biases the output up or down.
- Since we start with random weights anyways, can ignore the $-\theta$ notion, and just think of the bias as an extra available weight. (note the author uses a -1 input)
- Always use a bias weight

Perceptron Rule Example

- Assume a 3 input perceptron plus bias (it outputs 1 if net > 0, else 0)
- Assume a learning rate c of 1 and initial weights all 0: $\Delta w_i = c(t - z)\, x_i$
- Training set 0 0 1 -> 0
  1 1 1 -> 1
  1 0 1 -> 1
  0 1 1 -> 0

Pattern	Target (t)	Weight Vector (w_i)	Net	Output (z)	ΔW
0 0 1 1	0	0 0 0 0			

Example

- Assume a 3 input perceptron plus bias (it outputs 1 if net > 0, else 0)
- Assume a learning rate c of 1 and initial weights all 0: $\Delta w_i = c(t - z)\, x_i$
- Training set
  0 0 1 -> 0
  1 1 1 -> 1
  1 0 1 -> 1
  0 1 1 -> 0

Pattern	Target (t)	Weight Vector (w_i)	Net	Output (z)	ΔW
0 0 1 1	0	0 0 0 0	0	0	0 0 0 0
1 1 1 1	1	0 0 0 0			

Example

- Assume a 3 input perceptron plus bias (it outputs 1 if net > 0, else 0)
- Assume a learning rate c of 1 and initial weights all 0: $\Delta w_i = c(t - z)\, x_i$
- Training set 0 0 1 -> 0
  1 1 1 -> 1
  1 0 1 -> 1
  0 1 1 -> 0

Pattern	Target (t)	Weight Vector (w_i)	Net	Output (z)	ΔW
0 0 1 1	0	0 0 0 0	0	0	0 0 0 0
1 1 1 1	1	0 0 0 0	0	0	1 1 1 1
1 0 1 1	1	1 1 1 1			

Peer Instruction

- I pose a *challenge question* (often multiple choice), which will help solidify understanding of topics we have studied
  - Might not just be one correct answer
- You each get some time alone (1-2 minutes) to come up with your answer and vote – use Mentimeter (anonymous)
- Then you get some time to convince your group (neighbors) why you think you are right (2-3 minutes)
  - Learn from and teach each other!
- You vote again. May change your vote if you want.
- We discuss together the different responses, show the votes, give you opportunity to justify your thinking, and give you further insights

Peer Instruction (PI) Why

- Studies show this approach improves learning
- Learn by doing, discussing, and teaching each other
  - Curse of knowledge/expert blind-spot
  - Compared to talking with a peer who just figured it out and who can explain it in your own jargon
  - You never really know something until you can teach it to someone else – More improved learning!
- Learn to reason about your thinking and answers
- More enjoyable - You are involved and active in the learning

How Groups Interact

- Best if group members have different initial answers
- 3 is the "magic" group number
  - You can self-organize "on-the-fly" or sit together specifically to be a group
  - Can go 2-4 on a given day to make sure everyone is involved
- Teach and learn from each other: Discuss, reason, articulate
- If you know the answer, listen to where colleagues are coming from first, then be a great humble teacher, you will also learn by doing that, and you'll be on the other side in the future
  - I can't do that as well because every small group has different misunderstandings and you get to focus on your particular questions
- Be ready to justify to the class your vote and justifications!

Challenge Question - Perceptron

- Assume a 3 input perceptron plus bias (it outputs 1 if net > 0, else 0)
- Assume a learning rate c of 1 and initial weights all 0: $\Delta w_i = c(t - z)\, x_i$
- Training set 0 0 1 -> 0
  1 1 1 -> 1
  1 0 1 -> 1
  0 1 1 -> 0

Pattern	Target (t)	Weight Vector (w_i)	Net	Output (z)	ΔW
0 0 1 1	0	0 0 0 0	0	0	0 0 0 0
1 1 1 1	1	0 0 0 0	0	0	1 1 1 1
1 0 1 1	1	1 1 1 1			

- Once it converges the final weight vector will be
  A. 1 1 1 1
  B. -1 0 1 0
  C. 0 0 0 0
  D. 1 0 0 0
  E. None of the above

Example

- Assume a 3 input perceptron plus bias (it outputs 1 if net > 0, else 0)
- Assume a learning rate c of 1 and initial weights all 0: $\Delta w_i = c(t - z)\, x_i$
- Training set 0 0 1 -> 0
  1 1 1 -> 1
  1 0 1 -> 1
  0 1 1 -> 0

Pattern	Target (t)	Weight Vector (w_i)	Net	Output (z)	ΔW
0 0 1 1	0	0 0 0 0	0	0	0 0 0 0
1 1 1 1	1	0 0 0 0	0	0	1 1 1 1
1 0 1 1	1	1 1 1 1	3	1	0 0 0 0
0 1 1 1	0	1 1 1 1			

Example

- Assume a 3 input perceptron plus bias (it outputs 1 if net > 0, else 0)
- Assume a learning rate c of 1 and initial weights all 0: $\Delta w_i = c(t - z)\, x_i$
- Training set 0 0 1 -> 0
  1 1 1 -> 1
  1 0 1 -> 1
  0 1 1 -> 0

Pattern	Target (t)	Weight Vector (w_i)	Net	Output (z)	ΔW
0 0 1 1	0	0 0 0 0	0	0	0 0 0 0
1 1 1 1	1	0 0 0 0	0	0	1 1 1 1
1 0 1 1	1	1 1 1 1	3	1	0 0 0 0
0 1 1 1	0	1 1 1 1	3	1	0 -1 -1 -1
0 0 1 1	0	1 0 0 0			

Example

- Assume a 3 input perceptron plus bias (it outputs 1 if net > 0, else 0)
- Assume a learning rate c of 1 and initial weights all 0: $\Delta w_i = c(t - z)\, x_i$
- Training set 0 0 1 -> 0
  1 1 1 -> 1
  1 0 1 -> 1
  0 1 1 -> 0

Pattern	Target (t)	Weight Vector (w_i)	Net	Output (z)	ΔW
0 0 1 1	0	0 0 0 0	0	0	0 0 0 0
1 1 1 1	1	0 0 0 0	0	0	1 1 1 1
1 0 1 1	1	1 1 1 1	3	1	0 0 0 0
0 1 1 1	0	1 1 1 1	3	1	0 -1 -1 -1
0 0 1 1	0	1 0 0 0	0	0	0 0 0 0
1 1 1 1	1	1 0 0 0	1	1	0 0 0 0
1 0 1 1	1	1 0 0 0	1	1	0 0 0 0
0 1 1 1	0	1 0 0 0	0	0	0 0 0 0

Perceptron Homework

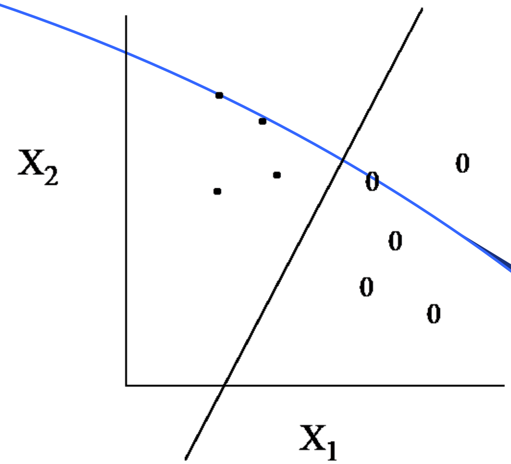
- Assume a 3 input perceptron plus bias (it outputs 1 if net > 0, else 0)
- Assume a learning rate c of 1 and initial weights all 1: $\Delta w_i = c(t - z)\, x_i$
- Show weights after each pattern for just one epoch
- Training set 1 0 1 -> 0
  1 .5 0 -> 0
  1 -.4 1 -> 1
  0 1 .5 -> 1

Pattern	Target (t)	Weight Vector (w_i)	Net	Output (z)	ΔW
		1 1 1 1			

Training Sets and Noise

- Assume a Probability of Error at each input and output value each time a pattern is trained on
- 0 0 1 0 1 1 0 0 1 1 0 -> 0 1 1 0
- i.e. P(error) = .05
- Or a probability that the algorithm is applied wrong (opposite) occasionally
- Averages out over learning

Linear Separability



2-d case (two inputs)

$$W_1X_1 + W_2X_2 > \theta \quad (Z=1)$$

$$W_1X_1 + W_2X_2 < \theta \quad (Z=0)$$

So, what is decision boundary?

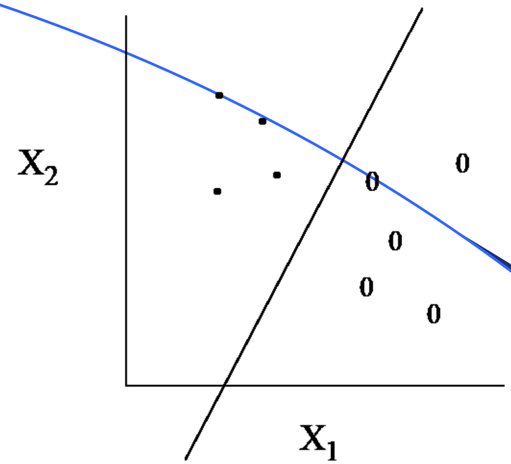
$$W_1X_1 + W_2X_2 = \theta$$

$$X_2 + W_1X_1/W_2 = \theta/W_2$$

$$X_2 = (-W_1/W_2)X_1 + \theta/W_2$$

$$Y = MX + B$$

Linear Separability



2-d case (two inputs)

If no bias weight, the hyperplane must go through the origin.
Note that since θ = -bias, the equation with bias is:
$X_2 = (-W_1/W_2)X_1 - bias/W_2$

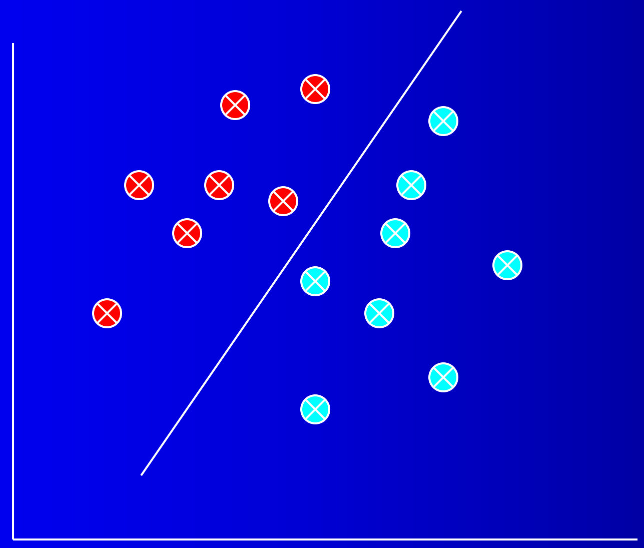
$$W_1X_1 + W_2X_2 > \theta \ (Z=1)$$
$$W_1X_1 + W_2X_2 < \theta \ (Z=0)$$

So, what is decision boundary?

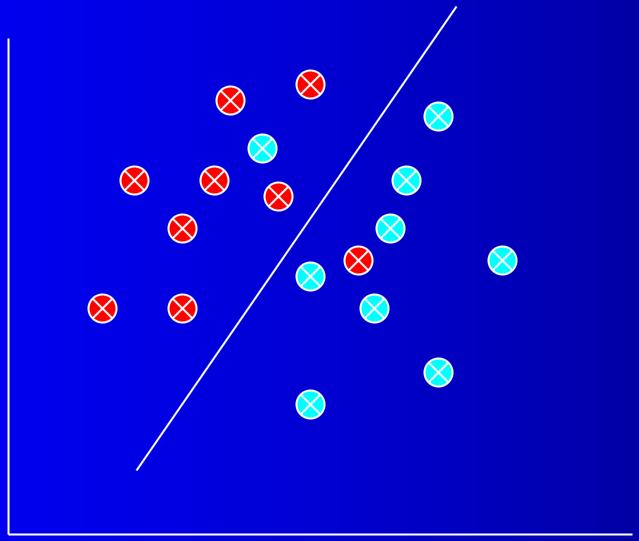
$$W_1X_1 + W_2X_2 = \theta$$
$$X_2 + W_1X_1/W_2 = \theta/W_2$$
$$X_2 = (-W_1/W_2)X_1 + \theta/W_2$$

$$Y = MX + B$$

Linear Separability

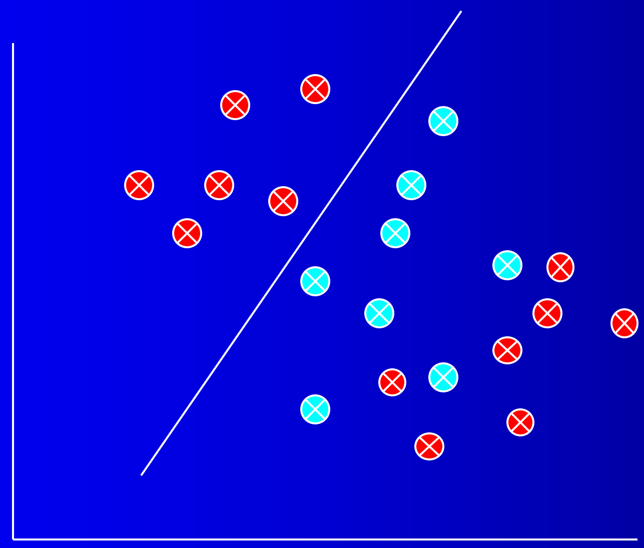


Linear Separability and Generalization



When is data noise vs. a legitimate exception

Limited Functionality of Hyperplane

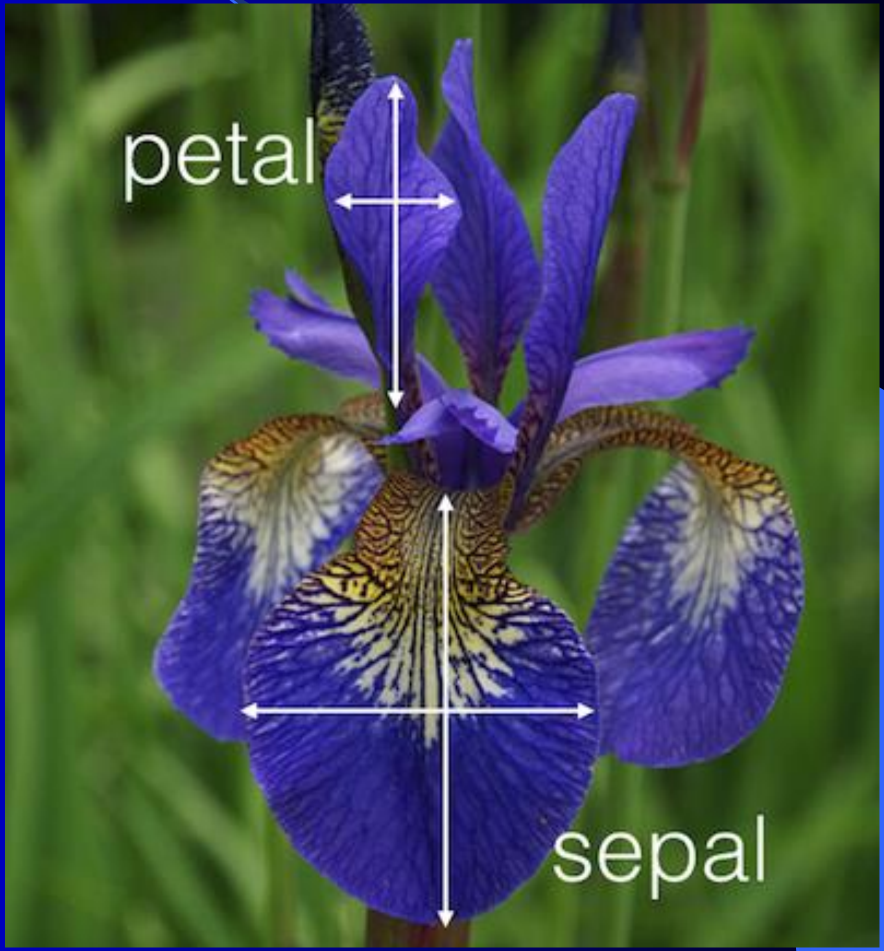


How to Handle Multi-Class Output

- This is an issue with learning models which only support binary classification (perceptron, SVM, etc.)
- Create 1 perceptron for each output class, where the training set considers all other classes to be negative examples (one vs the rest)
  - Run all perceptrons on novel data and set the output to the class of the perceptron which outputs high
  - If there is a tie, choose the perceptron with the highest net value
- Another approach: Create 1 perceptron for each pair of output classes, where the training set only contains examples from the 2 classes (one vs one)
  - Run all perceptrons on novel data and set the output to be the class with the most wins (votes) from the perceptrons
  - In case of a tie, use the net values to decide
  - Number of models grows by the square of the output classes

UC Irvine Machine Learning Data Base
Iris Data Set

4.8,3.0,1.4,0.3,	Iris-setosa
5.1,3.8,1.6,0.2,	Iris-setosa
4.6,3.2,1.4,0.2,	Iris-setosa
5.3,3.7,1.5,0.2,	Iris-setosa
5.0,3.3,1.4,0.2,	Iris-setosa
7.0,3.2,4.7,1.4,	Iris-versicolor
6.4,3.2,4.5,1.5,	Iris-versicolor
6.9,3.1,4.9,1.5,	Iris-versicolor
5.5,2.3,4.0,1.3,	Iris-versicolor
6.5,2.8,4.6,1.5,	Iris-versicolor
6.0,2.2,5.0,1.5,	Iris-viginica
6.9,3.2,5.7,2.3,	Iris-viginica
5.6,2.8,4.9,2.0,	Iris-viginica
7.7,2.8,6.7,2.0,	Iris-viginica
6.3,2.7,4.9,1.8,	Iris-viginica



Objective Functions: Accuracy/Error

- How do we judge the quality of a particular model (e.g. Perceptron with a particular setting of weights)
- Consider how accurate the model is on the data set
  - *Classification accuracy* = # Correct/Total instances
  - *Classification error* = # Misclassified/Total instances (= 1 – acc)
- Usually minimize a Loss function (aka cost, error)
- For real valued outputs and/or targets
  - Pattern error = Target – output: Errors could cancel each other
    - $\Sigma|t_j - z_j|$ (L1 loss), where j indexes all outputs in the pattern
    - Common approach is *Squared Error* = $\Sigma(t_j - z_j)^2$ (L2 loss)
  - Sum squared error (SSE) = Σ pattern squared errors = $\Sigma\,\Sigma\,(t_{ij} - z_{ij})^2$ where i indexes all the patterns in training set
- For nominal data, pattern error is typically 1 for a mismatch and 0 for a match
  - For nominal (including binary) output and targets, L1, L2, and classification error are equivalent

Mean Squared Error

- Mean Squared Error (MSE) – SSE/*n* where *n* is the number of instances in the data set
  - This can be nice because it normalizes the error for data sets of different sizes
  - MSE is the average squared error per pattern
- Root Mean Squared Error (RMSE) – is the square root of the MSE
  - This puts the error value back into the same units as the features and can thus be more intuitive
    - Since we squared the error on the SSE
  - RMSE is the average distance (error) of targets from the outputs in the same scale as the features
  - Note RMSE is the root of the total data set MSE, and NOT the sum of the root of each individual pattern MSE

Challenge Question - Error

- Given the following data set, what is the L1 ($\Sigma|t_i - z_i|$), SSE (L2) ($\Sigma(t_i - z_i)^2$), MSE, and RMSE error for the entire data set?

x	y	Output	Target	Data Set
2	-3	1	1	
0	1	0	1	
.5	.6	.8	.2	
L1				?
SSE				?
MSE				?
RMSE				?

A. .4 1 1 1
B. 1.6 2.36 1 1
C. .4 .64 .21 0.453
D. 1.6 1.36 .673 .82
E. None of the above

Challenge Question - Error

- Given the following data set, what is the L1 ($\Sigma|t_i - z_i|$), SSE (L2) ($\Sigma(t_i - z_i)^2$), MSE, and RMSE error for the entire data set?

x	y	Output	Target	Data Set
2	-3	1	1	
0	1	0	1	
.5	.6	.8	.2	
L1				1.6
SSE				1.36
MSE				1.36/3 = .453
RMSE				.453^.5 = .673

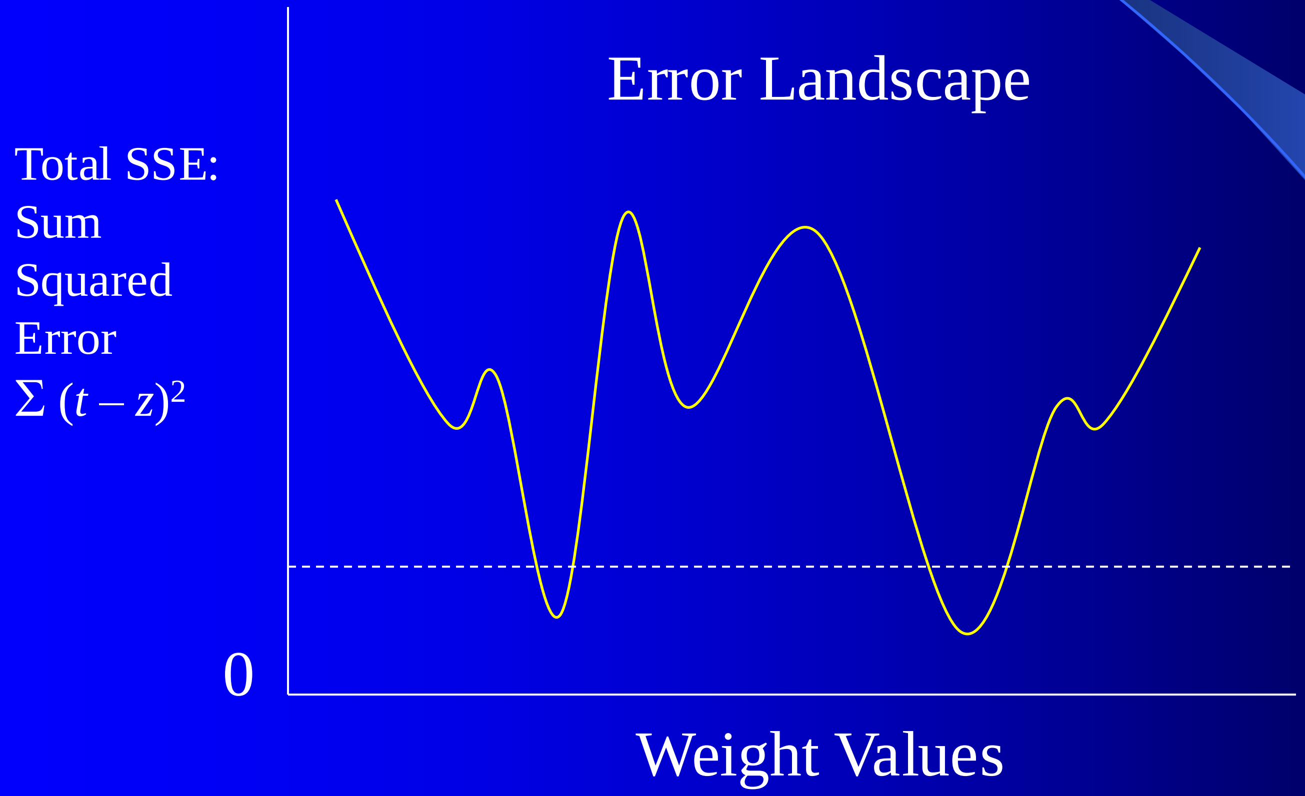
A. .4 1 1 1
B. 1.6 2.36 1 1
C. .4 .64 .21 0.453
D. 1.6 1.36 .673 .82
E. None of the above

Error Values Homework

- Given the following data set, what is the L1, SSE (L2), MSE, and RMSE error of Output1, Output2, and the entire data set? Fill in cells that have a ?.
  - Notes: For instance 1 the L1 pattern error is 1 + .4 = 1.4 and the SSE pattern error is 1 + .16 = 1.16. The Data Set L1 and SSE errors will just be the sum of each of the pattern errors.

Instance	x	y	Output1	Target1	Output2	Target 2	Data Set
1	-1	-1	0	1	.6	1.0	
2	-1	1	1	1	-.3	0	
3	1	-1	1	0	1.2	.5	
4	1	1	0	0	0	-.2	
L1			?		?		?
SSE			?		?		?
MSE			?		?		?
RMSE			?		?		?

Gradient Descent Learning: Minimize (Maximize) the Objective Function



Deriving a Gradient Descent Learning Algorithm

- Goal is to decrease overall error (or other loss function) each time a weight is changed
- Total Sum Squared error one possible loss function
  E: $\Sigma\,(t - z)^2$
- Seek a weight changing algorithm such that $\boxed{\dfrac{\partial E}{\partial w_{ij}}}$ is negative
- If a formula can be found then we have a gradient descent learning algorithm
- Delta rule is a variant of the perceptron rule which gives a gradient descent learning algorithm with perceptron nodes

Delta rule algorithm

- Delta rule uses (target - net) before the net value goes through the threshold in the learning rule to decide weight update

$$\Delta w_i = c(t - net)x_i$$

- Weights are updated even when the output would be correct
- Because this model is single layer and because of the SSE objective function, the error surface is guaranteed to be parabolic with only one minima
- Learning rate
  - If learning rate is too large can jump around global minimum
  - If too small, will get to minimum, but will take a longer time
  - Can decrease learning rate over time to give higher speed and still attain the global minimum (although exact minimum is still just for training set and thus…)

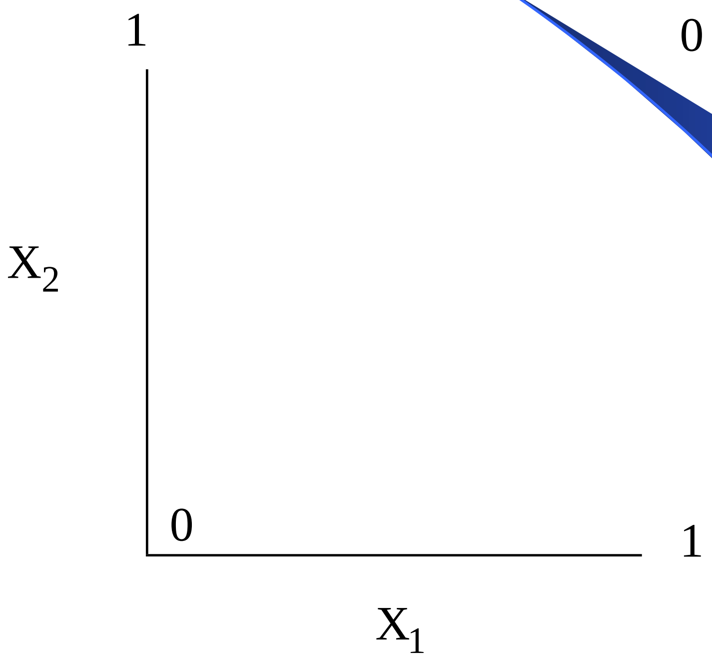
Batch vs Stochastic Update

- To get the true gradient with the delta rule, we need to sum errors over the entire training set and only update weights at the end of each epoch
- Batch (gradient) vs stochastic (on-line, incremental)
  - SGD (Stochastic Gradient Descent)
  - With the stochastic delta rule algorithm, you update after every pattern, just like with the perceptron algorithm (even though that means each change may not be along the true gradient)
  - Stochastic is more efficient and best to use in almost all cases, though not all have figured it out yet
  - We'll talk about this in more detail when we get to Backpropagation

Perceptron rule vs Delta rule

- Perceptron rule (target - thresholded output) guaranteed to converge to a separating hyperplane if the problem is linearly separable. Otherwise may not converge – could get in a cycle
- Singe layer Delta rule guaranteed to have only one global minimum. Thus, it will converge to the best SSE solution whether the problem is linearly separable or not.
  - Could have a higher misclassification rate than with the perceptron rule and a less intuitive decision surface – we will discuss this later with regression where Delta rules is more appropriate
- Stopping Criteria – For these models we stop when no longer making progress
  - When you have gone a few epochs with no significant improvement/change between epochs (including oscillations)

Exclusive Or



Is there a dividing hyperplane?

Linearly Separable Boolean Functions

- d = # of dimensions (i.e. inputs)

Linearly Separable Boolean Functions

- d = # of dimensions
- $P = 2^d$ = # of Patterns

Linearly Separable Boolean Functions

- d = # of dimensions
- $P = 2^d$ = # of Patterns
- $2^P = 2^{2^d}$ = # of Functions

n	Total Functions	Linearly Separable Functions
0	2	2
1	4	4
2	16	14

Linearly Separable Boolean Functions

- d = # of dimensions
- $P = 2^d$ = # of Patterns
- $2^P = 2^{2^d}$ = # of Functions

n	Total Functions	Linearly Separable Functions
0	2	2
1	4	4
2	16	14
3	256	104
4	65536	1882
5	4.3×10^9	94572
6	1.8×10^{19}	1.5×10^7
7	3.4×10^{38}	8.4×10^9

Linearly Separable Functions

$$LS(P,d) = 2 \sum_{i=0}^{d} \frac{(P-1)!}{(P-1-i)!i!} \quad for\ P > d$$

$$= 2^P \quad for\ P \leq d$$

(All patterns for d=P)

i.e. all 8 ways of dividing 3 vertices of a cube for d=P=3

Where *P* is the # of patterns for training and *d* is the # of inputs

$$\lim_{d \to \infty} (\#\ of\ LS\ functions) = \infty$$

Linear Models which are Non-Linear in the Input Space

- So far we have used $f(\mathbf{x},\mathbf{w}) = sign(\sum_{1=1}^{n} w_i x_i)$
- We could preprocess the inputs in a non-linear way and do

$$f(\mathbf{x},\mathbf{w}) = sign(\sum_{1=1}^{m} w_i \phi_i(\mathbf{x}))$$

- To the perceptron it looks the same but with more/different inputs. It still uses the same learning algorithm.
- For example, for a problem with two inputs x and y (plus the bias), we could also add the inputs x^2, y^2, and $x \cdot y$
- The perceptron would just consider it is a 5-dimensional task, and it is linear (5-d hyperplane) in those 5 dimensions
  – But what kind of decision surfaces would it allow for the original 2-*d* input space?

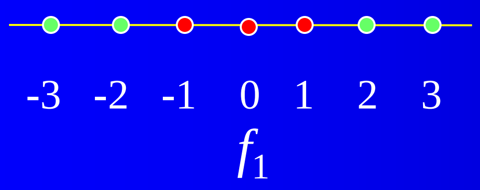
Quadric Machine

- All quadratic surfaces (2nd order)
  - ellipsoid
  - parabola
  - etc.
- That significantly increases the number of problems that can be solved
- Can we solve XOR with this model?

Quadric Machine

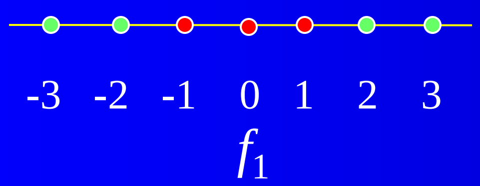
- All quadratic surfaces (2^{nd} order)
  - ellipsoid
  - parabola
  - etc.
- That significantly increases the number of problems that can be solved
- But still many problem which are not quadrically separable
- Could go to 3^{rd} and higher order features (cubic), but number of possible features grows exponentially
- Multi-layer neural networks will allow us to discover high-order features automatically from the input space

Simple Quadric Example



- What is the decision surface for a 1-d (1 input) problem?
- Perceptron with just feature f_1 cannot separate the data
- Could we add a transformed feature to our perceptron?

Simple Quadric Example



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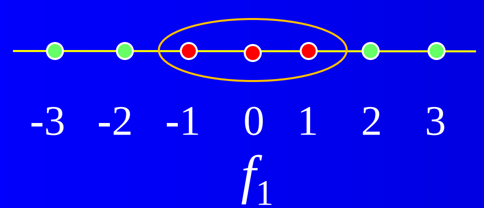
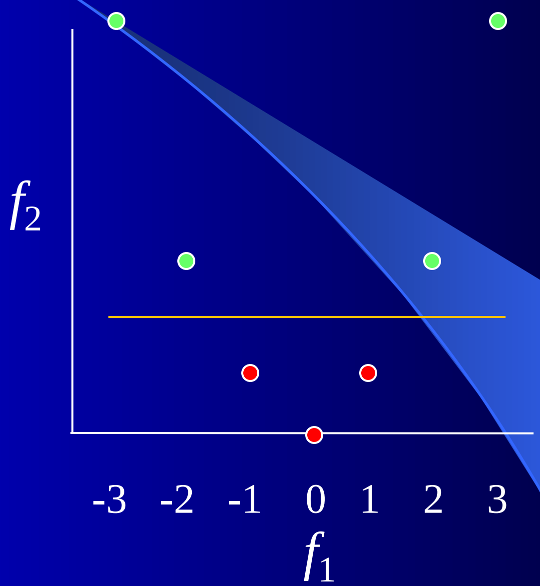
- Could we add a transformed feature to our perceptron?
- $f_2 = f_1^2$

Simple Quadric Example





- Perceptron with just feature f_1 cannot separate the data
- Could we add another feature to our perceptron $f_2 = f_1^2$
- Note could also think of this as just using feature f_1 but now allowing a quadric surface to divide the data
  - Note that f_1 not actually needed in this case

Quadric Machine Homework

- Assume a 2-input perceptron expanded to be a quadric (2nd order) perceptron, with 5 inputs/weights (x, y, $x \cdot y$, x^2, y^2) and the bias weight
  - Assume it outputs 1 if net > 0, else 0
- Assume a learning rate c of .5 and initial weights all 0
  - $\Delta w_i = c(t - z)\, x_i$
- Show all weights after each pattern for one epoch with the following training set

x	y	Target
0	.4	0
-.1	1.2	1
.5	.8	0