where

$$\langle a \rangle = 2^{-d} \sum_{x=0}^{2^d} a_x$$

It has been shown [15] that after $T = O(\sqrt{N})$ ($N = 2^d$) iterations the amplitude a_p becomes very close to unity while the amplitudes of the other states almost vanish. At this point, measuring the state of *NameFinder-in-Database* will, with high probability, produce that basis state $| p \rangle$ and the entangled name coded in the remaining qubits can be directly retrieved.

Note that the database considered here is complete: it contains all possible 2^d phone numbers. So, the collapse of the system into any basis state will provide a valid database record. This situation corresponds to the absence of spurious memories in quantum models of associative memory.

3 Completing quantum associative memory

A quantum associative memory with a capacity exponential in the number of qubits and based on Grover's algorithm has been proposed by Ventura and Martinez [10-12]. This kind of memory solves the problem of *pattern completion*. It can restore the full pattern when presented with only a partial one. It is crucial that this partial pattern would exactly coincide with some part of a valid full pattern. Then, in the recall phase only the remainder of the pattern is reconstructed while the initially presented partial pattern remains intact. This kind of associative memory clearly differs from the general statement of associative search. Indeed, general associative memory should also retrieve valid memory items when presented with noisy versions of these patterns. In fact, the original Grover's algorithm also solves the completion problem – in the interpretation described above it retrieves a full pattern (*name, number*) when presented with a partial pattern (*?, number*).

The main difference between the quantum associative memory developed in [10-12] and Grover's search algorithm of an unsorted database is that for the quantum associative memory the number of entries is smaller than 2^d and they form a set M of so called *memory states*. This is an important distinction for at least two reasons. First, while the creation of a quantum superposition containing all 2^d possible basis states is a straightforward operation [9], the creation of a quantum superposition containing only those basis states that correspond to valid memories is a nontrivial task, the solution to which is detailed in [14]. Second, the superposition of all 2^d basis states is a special case of the general wave function and researchers have to this point had difficulty discovering a practical use for Grover's algorithm, theoretically spectacular as it is. The quantum associative memory of [10-12] provides perhaps the first such practical application.

Hence in the case of the quantum associative memory, the initial state of *NameFinder-in- Memory* is described as a superposition of these states

$$| \psi^{(0)} \rangle = | m \rangle = \sum_{x \in M} \frac{1}{\sqrt{P}} | x \rangle \tag{6}$$

where P denotes the number of memorized patterns.

Also, the transformation performed by *NameFinder-in- Memory* now suggests the inversion of the amplitudes around the average *of only the amplitudes of memorized patterns* (zero amplitudes of the other ones play no role) [10]:

2'. $U_s \Rightarrow U_m$: *NameFinder-in- Memory* inverts amplitudes of all states around the average value for memorized patterns.

$$a_x \to 2\langle a \rangle_M - a_x \tag{7}$$

$$\langle a \rangle_M = \frac{1}{P} \sum_{x \in M} a_x$$

(Note, however, that in other versions of the model the original transformation U_s is used [11-12]). Further, it is assumed that *Oracle* knows *a part of one of the memorized patterns* and no other pattern has the same part. Then Grover's algorithm is used to find the pattern having this part and entanglement permits the restoration of the remainder of this pattern.

3.1 Completing versus general associative memory

In general, associative search suggests the possibility of retrieving valid memory items when presented with any possible external stimulus, including noisy stimuli. It is desirable to retrieve the memory state which is most similar to the given stimulus, i.e. that memory state which differs from the stimulus in the minimal number of bits (qubits). In the binary case, this corresponds to the minimizing the Hamming distance between query and memory state. So, this kind of memory is a *correcting* memory rather than a *completing* one. However, the quantum associative memory proposed in [10-12] does not actually take into account the distance between states but only uses information about the presence of some prescribed bit values in the memory state. Moreover, the quantum state describing the simplest form of completing memory [10] will not evolve while performing iterations if the query (part of memory item) presented by *Oracle* does not coincide with some part of a valid memory state. In order to overcome these difficulties, it is possible to introduce a metric into the quantum search algorithm in the form of *distributed queries*. Further, this complication of a query permits a quantum state to evolve in the case of the use of the transformation U_m; however it also leads to the appearance of spurious memories. Hence, while the completing associative memory proposed by Ventura and Martinez *can* be free from spurious memories (and can also contain them in other model variants [11-12]), general correcting associative memory should contain spurious states. Really, the appearance of spurious states in the model proposed in [12] is due to the traditional form of the transformation U_s. In a quantum associative memory of the general form discussed here, spurious memories arise even in the case when the transformation U_m is used. (Tables 2 and 3 illustrate the difference between these two situations.) The following discussion will be restricted to considering the case of the use of the transformation U_m in constructing a scheme for a general quantum associative memory. More complex cases will be considered in the future.

Distributed query means a query, or stimulus to the system, has the form of a superposition, just as a quantum memory does,

$$|b^p\rangle = \sum_{x=0}^{2^d-1} b_x^p \, |x\rangle \tag{8}$$

which in general includes *all* basis states. (Index *p* marks one of these states, $|p\rangle$, which plays the role of the *center* of the distribution). However, the introduction of distributed query demands the modification of the memory (phone book) in such a manner that it has *every* possible phone number (basis state), despite the fact that most of them have no corresponding name.

Table 2. A c*ompleting* associative memory *can* use a database (phone book) free from spurious memory (it contains only some of the possible numbers) and any query must represent an *exact* part of a phone number in the database, for example (*11), (01*) etc. Any query of a form similar to (10*) or (001) etc. will not cause any evolution of the quantum state describing a completing memory if transformation U_m is used.

Record	Name	Phone
1	Alice	010

2	Bob	111

Table 3. The g*eneral* (correcting) form of associative memory which uses distributed queries suggests that the database (phone book) includes a full set of numbers (8) but that some of them are not used, i.e. correspond to spurious memories. Collapse of the wave function into one of the basis states corresponding to such a spurious memory does not provide useful information.

Record	Name	Phone
1	not used	000
2	not used	001
3	Alice	010
4	not used	011
5	not used	100
6	not used	101
7	not used	110
8	Bob	111

This modification introduces the possibility of the quantum state collapsing into a basis state corresponding to an entry having the code "*not used*" in the *Name* field. In other words, such a memory will have so-called spurious memory states. However, before considering such a generalized associative memory, the application of Grover' algorithm to the case when *Oracle* defines not a single query (*marked state*) or finite set of such states [15] but rather defines a distributed *fuzzy* query.

First, suppose that in distributed query (8) real amplitudes are distributed such that the maximal value occurs for some definite state $|x\rangle = |p\rangle : |b_p^p|^2 = \max_x |b_x^p|^2$, and the amplitudes of the other basis states decrease monotonically with Hamming distance $|x-p| : |x-p| \uparrow \Rightarrow |b_x^p|^2 \downarrow$. From here on, $|p\rangle$ shall be referred to as the *query center*.

One way to satisfy these conditions follows from the binomial distribution

$$|b_x^p|^2 = q^{|p-x|}(1-q)^{d-|p-x|} \tag{9}$$

where $|p-x|$ denotes Hamming distance between $|p\rangle$ and $|x\rangle$, d is the number of qubits needed to code a phone number and $0 < q < 1/2$ is an arbitrary value which tunes the width of the distribution. For example, $d = 2$; $q = ¼$; $|p\rangle = |11\rangle$, produces the following distributed query

$$|b^p\rangle = \frac{3}{4}|11\rangle + \frac{\sqrt{3}}{4}|01\rangle + \frac{\sqrt{3}}{4}|10\rangle + \frac{1}{4}|00\rangle$$

It is important to note once more that introducing a distributed query with Hamming distance-dependent amplitudes for the basis states incorporates a metric into the model which permits comparison of the similarity of the stimulus and the retrieved memory. This is a necessary condition for associative searching. For this type of query the transformation performed by *Oracle* will have the form

$$U_b = 1 - 2|b^p\rangle\langle b^p| \tag{10}$$

It is well known that in the case of the traditional Grover's algorithm the transformation performed by *NameFinder-in- Database* inverts the amplitudes of the basis states around their mean value

$$a_x \to 2\langle a\rangle - a_x \tag{11}$$

In the case of the simple *completing* associative memory [10], *only memory states* are used in building the transformation performed by a network

$$U_m = 2\,|\,m\rangle\langle m\,| - 1 \tag{12}$$

Correspondingly, this transformation inverts the amplitudes of the memory states (their equiprobable superposition ordinarily forms the initial quantum state) around the average value of *only these memories* (remember, no additional spurious memories arise).

In the case of a distributed query, the *Oracle* transformation is defined by

$$\begin{aligned} U_b &= 1 - 2\,|\,b^p\rangle\langle b^p\,| \\ U_b &: \psi \to \psi - 2\,|\,b^p\rangle\langle b^p\,|\,\psi\rangle \end{aligned} \tag{13}$$

Since

$$\langle b^p\,|\,\psi\rangle = \sum_x b_x^p \langle x\,|\sum_y a_y\,|\,y\rangle = \sum_x b_x^p \sum_y a_y \langle x\,|\,y\rangle = \sum_x b_x^p \sum_y a_y \delta_{xy} = \sum_x a_x b_x^p \equiv \langle a\,|\,b^p\rangle \tag{14}$$

where δ is the Kronecker delta function and $\langle a\,|\,b^p\rangle$ represents an overlapping of the current quantum state with the query state,

$$\psi - 2\,|\,b^p\rangle\langle b^p\,|\,\psi\rangle = \sum_x a_x\,|\,x\rangle - 2\,|\,b^p\rangle\langle a\,|\,b^p\rangle = \sum_x \left(a_x - 2\langle a\,|\,b^p\rangle b_x^p\right)|\,x\rangle$$

or

$$a_x \to a_x - 2\langle a\,|\,b^p\rangle b_x^p \tag{15}$$

Note, that expression (15) can be transformed into the classical Grover's transformation U_s if $b_x^p = \delta_{xp}$. In this case $\langle a\,|\,b^p\rangle = \sum_x a_x b_x^p = \sum_x a_x \delta_{xp} = a_p$ and (15) takes a familiar form:

$$a_x \to a_x - 2a_p\delta_{xp}\,,\ \text{ i.e. }\ a_x = \begin{cases} a_x & \text{if } x \neq p \\ -a_x & \text{otherwise} \end{cases}$$

4 Grover's search in unsorted database with distributed query

Before considering the quantum associative memory with distributed queries, a generalization of Grover's original algorithm in the context of distributed queries will be presented. In general, a quantum database can have an arbitrary initial quantum state. In fact, the algorithm proposed in [14] is used to create database states representing quantum memory of arbitrary sets of patterns. How will Grover's original algorithm perform with such quantum states using a distributed query model?

4.1 Deriving the equations for averages

If for some iteration τ the state of the system is described by the superposition

$$\psi^{(\tau)} = \sum_x a_x^{(\tau)} \,| x > \tag{16}$$

then after the transformation U_b (first sub-step of an iteration), the superposition becomes

$$\psi^{(\tau+1/2)} = \sum_x a_x^{(\tau+1/2)} \,| x\rangle = \sum_x \left(a_x^{(\tau)} - 2\langle a \,| b^p\rangle^{(\tau)} b_x^p \right) | x\rangle \tag{17}$$

After U_s, the transformation performed by the *NameFinder-in-Memory* (second sub-step of an iteration), which inverts amplitudes around their average value,

$$\psi^{(\tau+1)} = \sum_x (2\langle a^{(\tau+1/2)}\rangle - a_x^{(\tau+1/2)}) \,| x\rangle = \sum_x \left\{ 2\langle a_x^{(\tau)} - 2\langle a \,| b^p\rangle^{(\tau)} b_x^p \rangle - a_x^{(\tau)} + 2\langle a \,| b^p\rangle^{(\tau)} b_x^p \right\} | x\rangle \tag{18}$$

Thus, one iteration causes the following change of amplitudes:

$$a_x^{(\tau+1)} = 2\langle a_x^{(\tau)} - 2\langle a \,| b^p\rangle^{(\tau)} b_x^p \rangle - a_x^{(\tau)} + 2\langle a \,| b^p\rangle^{(\tau)} b_x^p \tag{19}$$

or, equivalently,

$$a_x^{(\tau+1)} = 2\langle a\rangle^{(\tau)} - 4\langle a \,| b^p\rangle^{(\tau)} \langle b^p\rangle - a_x^{(\tau)} + 2\langle a \,| b^p\rangle^{(\tau)} b_x^p \tag{20}$$

Note, that Grover's original iteration scheme follows from (20) if

$$\begin{aligned} \langle a \,| b^p\rangle &= \sum_x a_x b_x^p = \sum_x a_x \delta_{xp} = a_p \\ \langle b^p\rangle &= \frac{1}{2^d} \sum_x b_x^p = \frac{1}{2^d} \sum_x \delta_{xp} = \frac{1}{2^d} \end{aligned} \tag{21}$$

Then, $a_x^{(\tau+1)} = 2\langle a\rangle^{(\tau)} - 4N^{-1} a_p^{(\tau)} - a_p^{(\tau)} + 2a_p^{(\tau)} \delta_{xp}$,

$$a_x^{(\tau+1)} = \begin{cases} 2\langle a\rangle^{(\tau)} - \left(1 + 4N^{-1}\right) a_p^{(\tau)} & \text{if } x \neq p \\ 2\langle a\rangle^{(\tau)} + \left(1 - 4N^{-1}\right) a_p^{(\tau)} & \text{otherwise} \end{cases} \tag{22}$$

where $N = 2^d$ denotes all possible states for a register consisting of d qubits. The particular case of initial state $|\psi\rangle = \frac{1}{2}\left(| 00\rangle + | 01\rangle + | 10\rangle + | 11\rangle\right)$, means that $\langle a\rangle^{(0)} = 1/2$ and for any p

$$a_p^{(1)} = \begin{cases} 2 \cdot \frac{1}{2} - (1 + 4/4) \cdot \frac{1}{2} = 0 & \text{if } x \neq p \\ 2 \cdot \frac{1}{2} + (1 - 4/4) \cdot \frac{1}{2} = 1 & \text{otherwise} \end{cases}$$

and so follows the well-known result that for these conditions any query will transform the state of the system to that of the marked state (find the requested number in the phone book) after only one iteration.

Now, starting from Equation (20) it is possible to obtain a closed system of two equations for the average values $\langle a\rangle$ and $\langle a\,|\,b^p\rangle$ (the approach is analogous to one used in [16]). Multiplying this equation by N^{-1} and adding the terms corresponding to all basis states results in

$$\langle a\rangle^{(\tau+1)} = 2\langle a\rangle^{(\tau)} - 4\langle a\,|\,b^p\rangle^{(\tau)}\langle b^p\rangle - \langle a\rangle^{(\tau)} + 2\langle a\,|\,b^p\rangle^{(\tau)}\langle b^p\rangle$$

or, equivalently,

$$\langle a\rangle^{(\tau+1)} = \langle a\rangle^{(\tau)} - 2\langle a\,|\,b^p\rangle^{(\tau)}\langle b^p\rangle \tag{23}$$

Multiplying each of the Equations (20) by its corresponding b_x^p value and summing over x produces

$$\langle a\,|\,b^p\rangle^{(\tau+1)} = 2\langle a\rangle^{(\tau)} N\langle b^p\rangle - 4\langle a\,|\,b^p\rangle^{(\tau)} N\langle b^p\rangle^2 - \langle a\,|\,b^p\rangle^{(\tau)} + 2\langle a\,|\,b^p\rangle^{(\tau)}\sum_x (b_x^p)^2$$

Taking into account that vector $\mathbf{b}^p = (b_0^p,\ldots,b_N^p)$ has a unity norm simplifies the last expression as

$$\langle a\,|\,b^p\rangle^{(\tau+1)} = 2N\langle b^p\rangle\langle a\rangle^{(\tau)} + (1 - 4N\langle b^p\rangle^2)\langle a\,|\,b^p\rangle^{(\tau)} \tag{24}$$

Finally, a closed system for the averages can be written as follows:

$$\begin{cases} \langle a\rangle^{(\tau+1)} &= \langle a\rangle^{(\tau)} - 2\langle b^p\rangle\langle a\,|\,b^p\rangle^{(\tau)} \\ \langle a\,|\,b^p\rangle^{(\tau+1)} &= 2N\langle b^p\rangle\langle a\rangle^{(\tau)} + (1-4N\langle b^p\rangle^2)\langle a\,|\,b^p\rangle^{(\tau)} \end{cases} \tag{25}$$

In order to reduce this to Grover's original scheme, assign $\langle a\,|\,b^p\rangle = a_p$ and $\langle b^p\rangle = N^{-1}$. Then

$$\begin{cases} \langle a\rangle^{(\tau+1)} &= \langle a\rangle^{(\tau)} - 2a_p^{(\tau)}/N \\ a_p^{(\tau+1)} &= 2\langle a\rangle^{(\tau)} + (1-4/N)a_p^{(\tau)} \end{cases} \tag{27}$$

Thus, for the case of Grover's iterations the second equation in (25) is transformed into the equation for the amplitude of the marked state (*Oracle*'s query). In what follows, the convenient and more compact notation below will be used to represent the average values of system (25)

$$\alpha^{(\tau)} \overset{def}{=} \langle a\rangle^{(\tau)},\ \beta^{(\tau)} \overset{def}{=} \langle a\,|\,b^p\rangle^{(\tau)}$$

4.2 Solving the equations for averages

An analytical solution of the system for average values will now be derived. Rewriting (25) using the notation introduced above results in

$$\begin{cases} \alpha^{(\tau+1)} = \alpha^{(\tau)} - 2\langle b^p\rangle\beta^{(\tau)} \\ \beta^{(\tau+1)} = 2N\langle b^p\rangle\alpha^{(\tau)} + (1-4N\langle b^p\rangle^2)\beta^{(\tau)} \end{cases} \tag{28}$$

Rewriting the second equation in (28) as

$$\beta^{(\tau+1)} = 2N\langle b^p \rangle \left\{ \alpha^{(\tau)} - 2\langle b^p \rangle \beta^{(\tau)} \right\} + \beta^{(\tau)} \tag{29}$$

it may be seen that the expression in curly braces is equivalent to the right hand side of the first equation of (28). Hence,

$$\beta^{(\tau+1)} = 2N\langle b^p \rangle \alpha^{(\tau+1)} + \beta^{(\tau)} \tag{30}$$

Manipulating the last equation gives

$$\alpha^{(\tau+1)} = \frac{\beta^{(\tau+1)} - \beta^{(\tau)}}{2N\langle b^p \rangle} \tag{31}$$

Substituting expression (31) into the first equation of system (28) and using (30) and some algebra gives

$$\beta^{(\tau+1)} + \beta^{(\tau-1)} = 2\beta^{(\tau)}(1 - 2N\langle b^p \rangle^2) \tag{32}$$

Suppose that the solution of Equation (32) is of the form

$$\beta^{(\tau)} = B\cos(\omega\tau + \varphi) \tag{33}$$

Inserting this expression into Equation (32) and using some trigonometry, an expression for the frequency may be obtained

$$\cos\omega = 1 - 2N\langle b^p \rangle^2 \tag{35}$$

or, because $\cos\omega = 1 - 2\sin^2(\omega/2)$,

$$\omega = 2\arcsin(N\langle b^p \rangle) \tag{36}$$

Now using Equation (31) produces the analytical form of $\alpha^{(\tau)}$

$$\alpha^{(\tau)} = \frac{\beta^{(\tau)} - \beta^{(\tau-1)}}{2N\langle b^p \rangle} = B\frac{\cos(\omega\tau + \varphi) - \cos(\omega(\tau-1) + \varphi)}{2N\langle b^p \rangle} \tag{37}$$

The values of the constants B and φ can be found from the initial conditions:

$$\alpha^{(0)} = B\frac{\cos\varphi - \cos(\omega - \varphi)}{2N\langle b^p \rangle} \tag{38}$$

$$\beta^{(0)} = B\cos\varphi \tag{39}$$

It follows from (38) and (39), that

$$\alpha^{(0)} = \frac{\beta^{(0)}}{\cos\varphi}\frac{\cos\varphi - \cos(\omega-\varphi)}{2N\langle b^p\rangle} \tag{40}$$

and after some transformation

$$\tan\varphi = \frac{1}{\sin\omega}\left\{1-\cos\omega - 2N\frac{\alpha^{(0)}}{\beta^{(0)}}\langle b^p\rangle\right\} \tag{41}$$

The other constant is expressible in terms of the phase value

$$B = \frac{\beta^{(0)}}{\cos\varphi} \tag{42}$$

Note that the amplitude of every basis state, including those that have a zero amplitude in the initial state, can take non-zero values during the iteration process. This corresponds to the development of spurious memories.

Example 1. Consider an unsorted database with phone numbers encoded with two qubits (d=2) and the distributed query

$$|b^p\rangle = \frac{4}{5}|00\rangle + \frac{2}{5}|01\rangle + \frac{2}{5}|10\rangle + \frac{1}{5}|11\rangle$$

This distribution of basis state amplitudes is visually represented by the histogram in the top right corner of Figure 1. Grover's iterations should be continued until the average overlap $\beta^{(\tau)} = \left\langle a \middle| b^p \right\rangle^{(\tau)}$ reaches one of the values $\{\pm 1\}$. After this it is necessary to perform a measurement and the probability for the system to be found in a given basis state becomes the prescribed function of Hamming distance from this state to the query center. In this example, if the initial state of the database is an equiprobable superposition of all basis states, the value +1 is reached after three iterations.

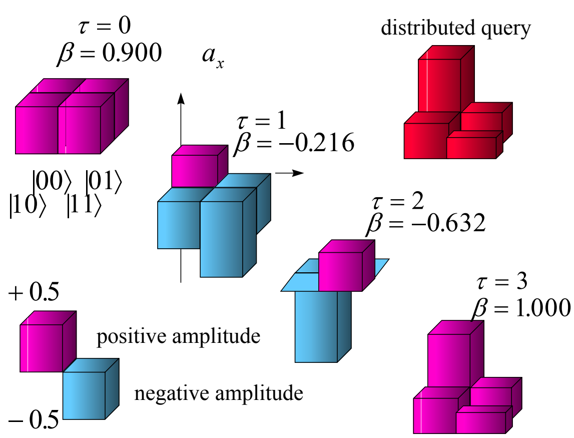

Figure 1. Right top corner: histogram of query amplitude distribution. Left top corner: initial equally weighted state, describing unsorted data base. Histograms of the iterated state amplitudes, with their corresponding β values, are placed along the diagonal. For the third iteration the distribution (lower right

corner) of the basis state amplitudes coincides with the distribution of amplitudes in the query (right top corner).

For the previous example the period of state oscillations can be found using expression (35). Since $N = 4$, and $\langle b^p \rangle = 1/4(0.8 + 0.4 + 0.4 + 0.2) = 0.45$, then

$$\cos\omega = 1 - 8\cdot(0.45\cdot 0.45) \cong -0.6;\ \omega \cong 0.7\pi;\ \frac{2\pi}{T} \cong 0.7\pi;\ T \cong 3$$

So, it may be concluded that the state of the system oscillates very quickly.

5 Quantum associative memory with distributed query

Now the general form for a quantum associative memory may be considered. Recall that if binary patterns are considered this kind of memory suggests the retrieval of a memory state whose Hamming distance from the presented stimulus is minimal. Also, recall that for the case of memory which includes only a restricted number of memory states it is necessary to change the transformation $U_s = 2|s\rangle\langle s| - 1$ to the transformation $U_m = 2|m\rangle\langle m| - 1$.

Of course, it would be desirable if any quantum state $|\psi\rangle$ describing our memory would not contain nonzero amplitudes for any basis state not corresponding to one of the memory patterns (the absence of spurious memories). But while this may be possible for a completing associative memory it is impossible in the case of distributed queries because after the transformation performed by *Oracle*

$$a_x \to a_x - 2\langle a | b^p \rangle b_x^p$$

so that in general all amplitudes take non-zero values during the course of the algorithm's iteration.

5.1 Model description

Hence, the state of our system will be described by an arbitrary wavefunction $|\psi\rangle$ and the transformation performed by the *NameFinder-in-Memory* will have the form

$$U_m : \psi \to 2|m\rangle\langle m|\psi\rangle - |\psi\rangle \tag{44}$$

Since

$$\langle m|\psi\rangle = \frac{1}{\sqrt{P}} \sum_{x\in M} \left(\langle x | \sum_y a_y | y \rangle \right) = \frac{1}{\sqrt{P}} \sum_{x\in M} \sum_y a_y \langle x | y \rangle = \frac{1}{\sqrt{P}} \sum_{x\in M} \sum_y a_y \delta_{xy} = \frac{1}{\sqrt{P}} \sum_{x\in M} a_x = \sqrt{P}\langle a \rangle_m$$

where P is the number of patterns in memory and $\langle a \rangle_m = \frac{1}{P} \sum_{x\in M} a_x$, it can be established that

$$2|m\rangle\langle m|\psi\rangle - |\psi\rangle = \sum_{x\in M} 2\langle a \rangle_m |x\rangle - \sum_x a_x |x\rangle$$

and the *NameFinder-in-Memory* transformation will be defined as

$$a_x \to \begin{cases} 2\langle a \rangle_m - a_x & \text{if } x \in M \\ -a_x & \text{otherwise} \end{cases} \tag{45}$$

Hence, the amplitude transformation performed by *NameFinder-in-Memory* will have a form similar to the *NameFinder-in-Database* transformation in Grover's algorithm, but this transformation will be applied only to the basis states corresponding to valid memories. For the rest of the basis states this transformation resembles the *Oracle* transformation (performing phase inversion). Finally, the generalized search algorithm described in the previous section can be adapted to the case in point, taking the following "anti-symmetrical" form:

$$\begin{array}{ll} \textit{Oracle}\text{ transformation:} & a_x \to a_x - 2\langle a \,|\, b^p \rangle b_x^p \\ \textit{NameFinder-in-Memory}\text{ transformation:} & a_x \to 2\langle a \,|\, m \rangle m_x - a_x \end{array} \qquad (46)$$

Example 2. Consider the case of a memory state containing a single valid pattern $|m\rangle = |01\rangle$ and the distributed query centered on the basis state $|11\rangle$,

$$|b\rangle = \frac{9}{10}|11\rangle + \frac{3}{10}|10\rangle + \frac{3}{10}|01\rangle + \frac{1}{10}|00\rangle$$

The *Oracle* transformation converts the initial state of the memory $|m\rangle^{(0)} = |01\rangle$ into

$$|m\rangle^{(1/2)} = -0.06|00\rangle + 0.82|01\rangle - 0.18|10\rangle - 0.54|11\rangle$$

and the *NameFinder-in-Memory* transformation completes a single iteration as

$$|m\rangle^{(1)} = 0.06|00\rangle + 0.82|01\rangle + 0.18|10\rangle + 0.54|11\rangle$$

Thus, after the first iteration the probability of measuring the system and finding the memory state $|01\rangle$ (which is a Hamming distance of one from the query center $|11\rangle$) takes a value $0.82^2 \approx 0.67$. The probability of collapsing into the spurious state $|11\rangle$ is $0.54^2 \approx 0.29$, and the probabilities for the system to be found in other spurious states are considerably lower.

Example 3. Consider the case of a memory containing two states $|m\rangle = \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|01\rangle$, and suppose that the query has the same form as for the previous example. Let the initial state of our memory be $|a\rangle^{(0)} = |m\rangle$. In this case after one iteration this state will take the form

$$|a\rangle^{(1)} = 0.54|00\rangle + 0.65|01\rangle + 0.17|10\rangle + 0.51|11\rangle$$

The probability for the system to be found after a measurement in the basis state $|01\rangle$, which is nearest to the query center (in the sense of Hamming distance) takes a value $0.65^2 \approx 0.42$. The probability for the system to be found in the memory state $|00\rangle$, for which query amplitude is minimal is not small: $0.54^2 \approx 0.29$. Also, the probability for the system to be observed in the spurious state $|11\rangle$, which corresponds to the query center, is fairly large. Both of these examples demonstrate that the properties of a quantum associative memory with distributed query seem to be reasonable.

In order to transform the expressions obtained earlier for Grover's algorithm with distributed query to the case of a general quantum associative memory for which the transformation *NameFinder-in-Memory* is used, it is necessary to make changes

$$\begin{gathered} \alpha = \langle a \rangle \Rightarrow \alpha_m = \langle a \,|\, m \rangle \\ \langle b^p \rangle \Rightarrow b_m = \langle b \,|\, m \rangle \\ 2N \Rightarrow 2 \end{gathered} \qquad (47)$$

For example , the expressions for the state frequency (35-36) take the form

$$\omega = \arccos(1 - 2b_m^2) \qquad (48)$$

$$\omega = 2\arcsin b_m \tag{49}$$

Now, using numbers from Example 3 gives $b_m = (0.1 + 0.3)/\sqrt{2} \cong 0.283$ and $\omega = \arccos(1 - 2\cdot 0.283^2) \cong 0.57$. Therefore $T = \frac{2\pi}{\omega} \cong 11$, and the memory recall will require 11 iterations of the algorithm to maximize the likelihood of obtaining the correct result (in this case observing the basis state $|01\rangle$).

5.2 Analytical solution for amplitudes

The third example will become clearer after deriving an analytical solution for the amplitudes. From Equations (46) it may be shown that

$$a_x^{(\tau+1)} = -a_x^{(\tau)} + 2\alpha_m m_{x_x} + 2\beta_m (b_x^p - 2b_m m_x) \tag{50}$$

Taking into account Equations (47) the following modifications of the expressions for the averages α_m and β_m originally written in expressions (33) and (37) may be obtained:

$$\alpha_m = \frac{B}{2b_m}(\cos(\omega\tau + \varphi) - \cos(\omega(\tau - 1) + \varphi)) = -\frac{B}{b_m}\sin(\omega\tau + \varphi - \omega/2)\sin(\omega/2) \tag{51}$$

$$\beta_m = B\cos(\omega\tau + \varphi) \tag{52}$$

Inserting the last expressions into Equation (50) results in

$$a_x^{(\tau+1)} = -a_x^{(\tau)} + 2B\left((b_x^p - 2b_m m_x)\cos(\omega\tau + \varphi) - \frac{m_x}{b_m}\sin(\omega/2)\sin(\omega\tau + \varphi - \omega/2)\right) \tag{53}$$

Suppose the solution of the last equation can be cast in the form

$$a_x = A_x \cos(\omega\tau + \delta_x) \tag{54}$$

Inserting (54) in (53) it is possible to derive

$$\begin{aligned} A_x(\cos(\omega\tau + \omega + \delta_x) + \cos(\omega\tau + \delta_x)) &\equiv 2A_x\cos(\omega/2)\cos(\omega\tau + \delta_x + \omega/2) = \\ 2B(b_x^p - 2b_m m_x)\cos(\omega\tau + \varphi) &- \frac{2Bm_x}{b_m}\sin(\omega/2)\sin(\omega\tau + \varphi - \omega/2) \equiv \\ &2B(f_x\cdot\cos(\omega\tau + \varphi) + g_x\cdot\sin(\omega\tau + \varphi)) \end{aligned} \tag{55}$$

where

$$f_x \overset{def}{=} \frac{m_x}{b_m}\sin^2(\omega/2) + b_x^p - 2b_m m_x \equiv b_x^p - m_x\sin(\omega/2) \tag{56}$$

$$g_x \stackrel{def}{=} \frac{m_x}{2b_m} \sin\omega \tag{57}$$

Introducing a variable

$$\zeta_x = \arccos\left(\frac{f_x}{\sqrt{f_x^2 + g_x^2}}\right) \tag{58}$$

Equation (55) can be transformed to

$$A_x \cos(\omega/2)\cos(\omega\tau + \delta_x + \omega/2) = B\sqrt{f_x^2 + g_x^2}\cos(\omega\tau + \varphi + \zeta_x) \tag{59}$$

From the last equation it follows immediately that the expressions for coefficients A_x and phases δ_x are

$$A_x = \frac{B\sqrt{f_x^2 + g_x^2}}{\cos(\omega/2)} \tag{60}$$

$$\delta_x = \zeta_x + \varphi - \omega/2 \tag{61}$$

Expressions (54, 60-61) give the analytical form of the amplitudes of the basis states for a general quantum associative memory based on the use of Grover's algorithm with distributed query. Figure 2 shows the analytical form of the amplitudes for the memory defined in the third example.

Figure 2. Dependence of basis state amplitudes on Grover's iterations for the memory defined in Example 3.

It can be seen in Figure 2 that the state $|01\rangle$ has a phase delay compared to the other valid memory state, $|00\rangle$. The observed delay is connected with the greater amplitude which basis state $|01\rangle$ has in the query. The change of phase δ_x due to the change of basis state amplitude b_x^p in a query is dependent only upon the variation of ζ_x; considering the derivative and applying some algebra

$$\frac{\partial}{\partial b_x^p}\cos\zeta_x = \frac{\partial}{\partial b_x^p}\left(\frac{b_x^p - m_x b_m}{\sqrt{(b_x^p)^2 - 2b_m b_x^p m_x + m_x^2}}\right) =$$

$$\frac{1}{\sqrt{(b_x^p)^2 - 2b_m b_x^p m_x + m_x^2}} - \frac{(b_x^p - m_x b_m)^2}{((b_x^p)^2 - 2b_m b_x^p m_x + m_x^2)^{3/2}} \equiv \frac{\sin^2\zeta_x}{\sqrt{(b_x^p)^2 - 2b_m b_x^p m_x + m_x^2}} \geq 0$$

Hence,

$$-\sin\zeta_x \frac{\partial\zeta_x}{\partial b_x^p} = -\frac{m_x \sin\omega}{2b_m}\cdot\frac{\zeta_x}{\partial b_x^p} \geq 0$$

or

$$\frac{\partial\zeta_x}{\partial b_x^p} \leq 0$$

Thus, the state $|01\rangle$ which has a greater amplitude in the query than does the state $|00\rangle$ will also have a lower phase value ζ_x, and, consequently, lower value of δ_x. In query formation, amplitudes of basis states monotonously decrease with Hamming distance from the query center; *therefore, the memory state nearest to this center will have maximal amplitude and, consequently, minimal phase value.* It may also be seen in Figure 2 that those basis states which do not belong to the set of valid memories *(spurious memories) all have the same phase value*.

As will be seen shortly, this fact is very important, and it can be verified using expression (58). Indeed, for any spurious memory, from (56-58) it follows that $m_x \equiv 0 \Rightarrow g_x \equiv 0 \Rightarrow \cos\zeta_x \equiv 1 \Rightarrow \zeta_x \equiv 0$. Then, from expression (61) the fact that $\delta_x = \varphi - \omega/2 = const$ may be derived.

Consider, once again, Figure 2. Maximal amplitude belongs to the state $|11\rangle$ for which the amplitude of the query is also maximal. On the other hand, the state $|00\rangle$, has both minimal amplitude and minimal query amplitude. However, in general, this relation is not valid. Indeed, the dependence of the basis state amplitude on query amplitude

$$A_x = \frac{B\sqrt{f_x^2 + g_x^2}}{\cos(\omega/2)}$$

can be rewritten using explicit expressions for the parameters f_x and g_x given in (56) and (57) as follows

$$A_x = \frac{B}{\cos(\omega/2)}\sqrt{(b_x^p)^2 - 2b_m m_x b_x^p + m_x^2}$$

The expression on the right hand side is not a monotonic function of b_x^p and takes a minimal value on an internal point on the interval [0,1]. Therefore, basis states with lower query amplitude can have greater amplitude in memory.

Note, that in principle, in a distributed query some amplitudes can have zero values. Then, corresponding basis states will never arise as spurious memories. Thus, spurious memories are generated by the query itself demonstrating the principle: *arise if suggested*.

The evolution of the amplitudes of memory states is rather complicated. In general there exists no *a priori* knowledge about the structure of memory nor about the correspondence between the location of memories and the query center in configuration space. It is therefore difficult to obtain an analytical expression for the number of iterations needed to reach the maximal values of amplitudes of memories in the vicinity of this center. It is clear that the difficulty in deriving the necessary estimate is connected with the probabilistic character of the parameter b_m. But the situation can be improved considerably by changing the way the memory is structured, taking advantage of the fact that oscillating spurious memories all have the same phase value.

5.3 Memories become easily retrieved when they become spurious

The trick is simply to exchange the valid states to be memorized with other (spurious) states and vice versa. Namely, create a memory whose initial state is

$$|\psi^{(0)}\rangle = |\widetilde{m}\rangle = \sum_{x \notin M} \frac{1}{\sqrt{P}} |x\rangle$$

and correspondingly, the transformation performed by this memory is defined as

$$U_{\widetilde{m}} : \psi = 2|\widetilde{m}\rangle\langle\widetilde{m}|\psi\rangle - |\psi\rangle$$

In some sense this kind of memory is ideologically similar to the immune system, which includes antibodies corresponding to antigens that *do not belong* to the host organism and has almost no antibodies to its own proteins. Analogically, the associative memory is formed by memorizing patterns which *should not be recalled*.

In this system , all valid memories be treated as spurious ones. All these memories will be in phase with a phase value of $-\pi/2$ and have initial amplitudes of zero. Therefore, the amplitudes of these states will evolve according to

$$a_x^\tau = A_x \sin \omega\tau$$

and because for these states $m_x = 0$,

$$A_x = \frac{B}{\cos(\omega/2)} \sqrt{(b_x^p)^2 - 2b_m m_x b_x^p + m_x^2} = \frac{B b_x^p}{\cos(\omega/2)}$$

based upon the query amplitudes. Applying this trick to Example 3, the initial state of memory is

$$b_m = 2^{-d/2} \sum_{x \notin M} b_x^p \cong \widetilde{b}_m = 2^{-d/2} \sum_{x=0}^{2^d-1} b_x^p$$

$$|\tilde{m}\rangle = \frac{1}{\sqrt{2}}|10\rangle + \frac{1}{\sqrt{2}}|11\rangle$$

Then, $\langle b\,|\,m\rangle = (0.3+0.9)/\sqrt{2} \cong 0.85$ and $\omega \cong 2.03$, and now the states of the system will oscillate with considerably lower (compared to the original 11) period $T = 2\pi/\omega \cong 3.1$.

Since the amplitudes of all spurious memories achieve their maximal values at $\tau = T/4$, the system will be in a state for which the amplitudes of these "spurious" memories become proportional to their amplitudes in the query for $\tau \cong 0.77$. Of course, the number of iterations must be integer valued so it is necessary to add some periods for $\tau + nT$ to be as near as possible to an integer value. In this case, for $n = 2$, 0.77 + 6.2 = 6.97 ≈ 7, so it will suffice to perform seven iterations of the algorithm before performing a measurement. After these iterations the state of system will be

$$|\psi\rangle = 0.19\,|00\rangle + 0.57\,|01\rangle + 0.6\,|10\rangle + 0.53\,|11\rangle$$

It is evident that the amplitudes of the "spurious" memories (really the valid memories) in this state are in the same proportion as in the query (a ratio of 1/3). Nevertheless, in general it seems that the difficulty inherent in the previous scheme still remains: a dependence of the frequency ω on the set of valid memory patterns.

However, taking into account that now the number of "valid" patterns is very near 2^d (because $p << 2^d$), $b_m = 2^{-d/2}\sum_{x \notin M} b_x^p \cong \tilde{b}_m = 2^{-d/2}\sum_{x=0}^{2^d-1} b_x^p$. Therefore, in contrast to the more intuitive approach (memorizing the valid patterns) the value of b_m can be approximated as *a priori* knowledge, dependent only on the form of the query, not on the form of the set of memory states. Therefore, using this approximated value of b_m and the corresponding approximation of the frequency $\tilde{\omega} = 2\arcsin\tilde{b}_m$, the number of iterations $T_{\max}$ needed to transform the system into a state such that the amplitudes of spurious (actually valid) memories become maximal and proportional to their amplitudes in the distributed query is

$$\tilde{\omega}T_{\max} = \frac{\pi}{2} \Rightarrow T_{\max} \approx \frac{\pi}{2\tilde{\omega}}$$

4.4. A note on initializing quantum states

It has been recently pointed out that there is potentially an inherent problem with current quantum computational algorithms [17]. Quantum computers and quantum algorithms rely heavily on the phase information of quantum states -- if the relative phases of the various states in a system are not correct, the computation will not work. Kak discusses the fact that quantum systems can possess random initial phases, whereas quantum algorithms implicitly assume some known initial phase conditions from which to begin the computation. The consensus seems to be that it is possible that this initial variability in the state phases may be compensated for by quantum error correction schemes [18][19]. However, these schemes may also be flawed. Classical error correction is based upon the fact that errors in classical systems are discrete -- a bit is flipped with some small probability. However, because quantum computational systems contain phase information, they are susceptible to a continuum of possible errors, and quantum error correction schemes developed to date address only a small number of special cases. Therefore, the issue to be resolved is whether or not in practice (that is in constructing a quantum computer) we will encounter mostly those few cases of error which have been treated in the literature or we will see the many other possibilities that Kak points out.

6 Conclusion

A model of quantum associative memory which is able to retrieve memory states with probability proportional to the amplitudes these states have in a query has been presented. This quantum memory can retrieve valid stored patterns from arbitrary stimuli represented by a distributed query of general form (fuzzy stimulus). Further investigation of the model is needed to estimate its other possible merits and limitations.

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