Recurrence Relations

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Adapted from appendix B of *Foundations of Algorithms* by Neapolitan and Naimipour.

1 Part I

1.1 Definitions and the Characteristic Function

A **recurrence relation** is a relation in which t_n is defined in terms of a smaller value of n. A recurrence does not define a unique function, but a recurrence together with some initial conditions does define a unique function. The **initial conditions** gives the values of t_i for some set of i's starting with $i = 0$. def. def.

A **solution** to a recurrence relation for a given set of initial conditions is an explicit function that defines t_n without using any t_i in the definition of t_n. def.

For **example** , the relation ex.

$$t_n = t_{n-1} + 1$$

is a recurrence relation and if the initial condition is

$$t_0 = j$$

for any value j, then the solution to $t_n = t_{n-1} + 1$ is

$$t_n = j + n$$

A **homogeneous linear recurrence equation with constant coefficients** is an equation of the form def.

$$a_0 t_n + a_1 t_{n-1} + ... + a_k t_{n-k} = 0$$

Let's go through that step by step.

- It's **linear** because every term t_k appears in the first power and term n depends on a sequence of $n - k$ terms for an integer k. def.
- Its **homogeneous** because it equals 0. def.
- and it has constant coefficients.

For **example** , let's solve the recurrence relation ex.

$$t_n - 5t_{n-1} + 6t_{n-2} = 0 \tag{1}$$

First off, note that its a homogeneous linear recurrence relation with constant coefficients.

Linear systems theory leads us to set

$$t_n = r^n$$

For some non-zero value of r. We don't know what r is, but we are going to require that the above equality holds. It remains to be proven that such an r exists. It also remains to be proven that replacing the nth term of t with the nth power of r results in the same solution to the recurrence relation. There are two ways to argue that replacing t_n with r^n gives a correct solution to the recurrence relation. First, we don't have to know beforehand that $t_n = r^n$ is going to work. After we find the roots of the characteristic function, we can verify that the substitution worked by testing the solution on some values. Second, we can prove that this substitution will work in general for any linear constant-coefficient homogeneous recurrence relation. That deeper answer is beyond the scope of the class but if you are interested, we recommend the text by Luenberger.[1]

Moving on, if we replace t_n with r^n in Equation 1 then we get the **characteristic function** for the recurrence relation def.

$$r^n - 5r^{n-1} + 6r^{n-2} = 0 \tag{2}$$

We can now see that setting $t_n = r^n$ is a solution to Equation 1 **if** r is a root of Equation 2.

Why is that? Since we have

$$t_n - 5t_{n-1} + 6t_{n-2} = r^n - 5r^{n-1} + 6r^{n-2} = 0$$

the roots of the polynomial in r are the values of r in which the polynomial equals zero. At those values of r, the values of the functions of both r and t_n work out to be zero–as required by the recurrence relation.

So what are the roots of

$$r^n - 5r^{n-1} + 6r^{n-2} = 0? \tag{3}$$

First, multiply the left side through by r^{n-2}/r^{n-2} to get

$$r^{n-2}\left(\frac{r^n}{r^{n-2}} - 5\frac{r^{n-1}}{r^{n-2}} + 6\frac{r^{n-2}}{r^{n-2}}\right) = 0$$

simplify the fractions in one step:

$$r^{n-2}\left(\frac{1}{r^{-2}} - 5\frac{1}{r^{-1}} + 6\frac{1}{1}\right) = 0$$

[1] David G. Luenberger, *Introduction to Dynamic Systems: Theory, Models, and Applications, 1979, John Wiley and Sons.*

Recall that $1/n^{-k} = n^k$, and we have

$$r^{n-2}(r^2 - 5r + 6) = 0$$

Then we can factor the polynomial into

$$r^{n-2}(r-2)(r-3) = 0$$

We assumed that r is non-zero, so we can divide by the first factor r^{n-2}, which leaves

$$(r-2)(r-3) = 0$$

Thus, the characteristic function (also: polynomial) remains on the left-hand side. Its roots are 2 and 3.

Now we go back and check that the roots are solutions to the recurrence relation. So we let

$$t_n = 3^n \text{ or } t_n = 2^n.$$

In this **example** , we let $t_n = 3^n$ and show that ex.

$$3^n - 5(3^{n-1}) + 6(3^{n-2}) = 0.$$

To do this, we first recall that

$$3^{n-2} = 3^{-1}3^{n-1} = \frac{1}{3}3^{n-1}$$

so that we can get

$$3^n - 5(3^{n-1}) + 2(3^{n-1}) = 0$$

which is

$$3^n - 3(3^{n-1}) = 0$$

and note again that

$$3^{n-1} = 3^{-1}3^n = \frac{1}{3}3^n$$

which leaves us with

$$3^n - 3^n = 0.$$

We have just verified that $t_n = 3^n$ is **a** solution for Equation 1. Note that we have not shown for which initial values 3^n is a solution, but we have shown that it is a solution. The second homework problem is to show that $t_n = 2^n$ is a solution as well. And the third homework problem is to show that any linear combination of 2^n and 3^n is a solution.

1.2 Finishing Linear Homogeneous Recurrences

First off, we need to wrap up solving linear homogeneous recurrences using the characteristic function. The final piece we need is a method for determining which values of c_1 and c_2 give solutions to a recurrence relation for a given set of initial values. The equation

$$t_n = c_1 3^n + c_2 2^n$$

is called the **general solution** to the recurrence. We want *the* solution for a given set of initial values. To do this, we simply plug in values of the initial conditions and solve for the c's. We can do this because we end up with n functions of n variables. def.

Returning to the **example** from the previous section, if we set ex.

$$t_0 = 0 \text{ and } t_1 = 1$$

then we can determine c_1 and c_2 by solving the system of equations

$$t_0 = c_1 3^0 + c_2 2^0 = 0$$

and

$$t_1 = c_1 3^1 + c_2 2^1 = 1$$

If you remember your linear algebra, then you will quickly see that the solution is $c_1 = 1$ and $c_2 = -1$. If you don't then here's a quick review (or tutorial). The basic idea is to pick a variable c_i, multiply one equation by a constant and add the equations together to eliminate c_i. You pick the constant carefully (basically you pick $-a_j/a_i$ where a_j is the coefficient on c_i in the other equation and a_i is the coefficient of c_i) so that the sum involving that term works out to 0.

Like so... Multiply the top equation by -3

$$\begin{aligned} c_1 3^0 + c_2 2^0 &= 0 \\ c_1 3^1 + c_2 2^1 &= 1 \end{aligned}$$

to get

$$\begin{aligned} -3c_1 + -3c_2 &= 0 \\ 3c_1 + 2c_2 &= 1 \end{aligned}$$

then add the equations and

$$\begin{array}{l} -3c_1 + -3c_2 = 0 \\ 3c_1 + 2c_2 = 1 \\ \hline -c_2 = 1 \end{array}$$

so $c_2 = -1$. Repeat the process, re-starting with the original system of equations to eliminate c_2 (hint: multiply the top equation by -2) and you find that $c_1 = 1$.

In the end, the solution is

$$t_n = 1(3^n) - 1(2^n) = 3^n - 2^n$$

if $t_0 = 0$ and $t_1 = 1$.

One last point. If we factor the characteristic function and find that one of the $(r - i)$ terms is duplicated j times, then i is a **root of multiplicity** j. For **example** , if we get the characteristic equation down to def. ex.

$$(r-1)(r-2)^3 = 0$$

then the characteristic equation has 2 as a root of multiplicity 3.

If a characteristic function has a root r of multiplicity j, then the following terms

$$t_n = r^n, t_n = nr^n, t_n = n^2r^n \text{ and } t_n = n^{j-1}r^n \tag{4}$$

must appear in the general solution (...multiplied by different constants in the sum of terms). This keeps each of the terms linearly independent, thus forming an appropriate basis for all solutions. For our example of

$$(r-1)(r-2)^3 = 0,$$

The general solution is

$$t_n = c_1 \cdot 1^n + c_2 \cdot 2^n + c_3 \cdot n2^n + c_4 \cdot n^2 2^n$$

1.3 Homework

1. Which of the following are homogeneous linear recurrences with constant coefficients?

$$6t_n + 1t_{n-1} + 4t_{n-2} = 0$$

$$43t_n + 5t_{n/2} + t_{n/4} = 0$$

$$t_n + 8t_n + 4t_{n-2} = 5$$

$$8t_n t_{n-2} + 51t_{n-1} + t_{n-2} = 0$$

2. Verify that $t_n = 2^n$ is a solution to Equation 1 (from the example on page 2). Note that we verified that $t_n = 3^n$ is a solution to Equation 1 earlier in this section.

3. Verify that $c_1 3^n + c_2 2^n$ is a solution for Equation 1 (also from the example on page 2) for any constants c_1 and c_2. The equation $c_1 3^n + c_2 2^n$ is called the general solution to a recurrence relation.

4. Solve $t_n = 4t_{n-1} - 3t_{n-2}$ for $n > 1$ with $t_0 = 0$ and $t_1 = 1$ using the characteristic function. (Hint: you have to convert it to a homogenous recurrence first. This is not hard.)

2 Part II

2.1 Nonhomogeneous Linear Recurrences

A **nonhomoegeneous linear recurrence** has the form def.

$$a_0t_n + a_1t_{n-1} + ... + a_kt_{n-k} = f(n)$$

where $f(n) \neq 0$. There is no known general solution technique for nonhomogeneous linear recurrences. So we will solve recurrences of the form

$$a_0t_n + a_1t_{n-1} + ... + a_kt_{n-k} = b^n p(n) \quad (5)$$

where $p(n)$ is a polynomial in n. The right-hand side is known as a **geometric forcing function**; "geometric" refers to the growth in the exponential term, def. and "forcing function" refers to the fact that the function provides the input to the system.

First, we will do an **example** using the brute force approach with $p(n) = 1$. ex. Solve the recurrence

$$t_n - 3t_{n-1} = 4^n$$

What we want are two versions of the recurrence that are equal to 4^{n-1}. Then we can subtract them and get a homogeneous recurrence. To get the first one, replace n with $n - 1$.

$$t_{n-1} - 3t_{n-2} = 4^{n-1}.$$

We can do that because the recurrence also defines t_{n-1} and t_{n-2} and so forth all the way down to t_{n-k+1}. To get the second one, divide the recurrence by 4.

$$\frac{t_n}{4} - \frac{3t_{n-1}}{4} = 4^{n-1}$$

Both of these recurrences have the same solution as the original recurrence. We subtract them to eliminate the 4^{n-1} term and get

$$\frac{t_n}{4} - \frac{7t_{n-1}}{4} + 3t_{n-2} = 0$$

which is a linear homogeneous recurrence that we already know how to solve.

In general, we have the following theorem which is a convenient shortcut to the brute force approach above: A nonhomogeneous linear recurrence of the form

$$a_0t_n + a_1t_{n-1} + ... + a_kt_{n-k} = b^n p(n)$$

can be transformed into a recurrence of the form

$$(a_0r^k + a_1r^{k-1} + ... + a_k)(r - b)^{d+1} = 0 \quad (6)$$

in which d is the order of the polynomial $p(n)$. (And the order of a polynomial is the largest exponent in the polynomial). The proof of this fact would follow the reasoning we used to homogenize the recurrence in the last example. We will omit the proof though. Note that the solution in Equation 6 has a root of multiplicity $d+1$.

For **example** , consider the following recurrence: ex.

$$t_n - 3t_{n-1} = 4^n(2n+1)$$

First, we reduce it to a homogeneous recurrence using Equation 6. To do that, we need to find k, d, b and the a_i's. k is 1 because t_{n-1} is the last term in the recurrence relation. d is 1 because the order of $(2n-1)$ is 1. b is 4 and $a_0 = 1$ with $a_k = a_1 = -3$. So we have the homogenous recurrence

$$(1r^1 - 3)(r-4)^{1+1} = 0 \tag{7}$$

which we know how to solve. The roots of Equation 7 are 3 and 4 and the root 4 has multiplicity 2. So the general solution is

$$t_n = c_1 3^n + c_2 4^n + c_3 n 4^n$$

which has three unknowns so we will need three initial values to get values for the three unknowns.

If we are given less than three initial values (which we could be because the original recurrence only needs one initial value), then we can get more initial values by computing them. For **example** , if we are given the initial value ex.

$$t_0 = 0$$

then we can compue t_1 as

$$t_1 - 3t_0 = 4^1(2(1)+1) \text{ is } t_1 - 3(0) = 4^1(2(1)+1)$$

and $t_1 = 12$.

2.2 Homework

1. Solve $t_n = 3t_{n-1} - 2t_{n-2} + n^2$ for $n > 1$ with $t_0 = 0$ and $t_1 = 1$ using the characteristic function. (Hint: Cast the recurrence relation into the form of equation 6 and solve it using the shortcut method.)

3 Part III

3.1 Change of variables (Domain Transformation)

Many of the recurrences that describe Divide and Conquer algorithms are not of the form that we can directly solve using the techniques we have studied thus far. These recurrences usually have the form:

$$t_n = l \cdot t_{n \div b} + g(n)$$

Where l and b are integers and $g(n) \in \Theta(n^k)$. The practice of writing the n as a subscript gets awkward at this point. It will now be useful to write recurrences using this equivalent notation:

$$t(n) = l \cdot t(n \div b) + g(n)$$

For example, the recurrence describing the complexity of merge sort (assuming n is a power of 2) is given as

$$t(n) = 2 \cdot t(\frac{1}{2} \cdot n) + n$$

The $t(\frac{1}{2} \cdot n)$ term causes the recurrence to not fit our general solution. Recall that $t(n)$ means the same thing as t_n. To solve this type of recurrence, we are going to use a **change of variables**. We demonstrate this through an **example** . Consider the recurrence def. ex.

$$\begin{aligned} T(n) &= T(\frac{n}{2}) + 1 \quad \text{for } n > 1, \text{ and } n \text{ a power of 2} \\ T(1) &= 1 \end{aligned}$$

To *change the variables* to a form we can solve, first, we replace n with 2^k (which we can do because we assumed that n was a power of 2). Second, we solve for k so that

$$n = 2^k, \quad \text{which means} \quad k = \lg n$$

Now substitute 2^k into the expression for n

$$\begin{aligned} T(2^k) &= T(\frac{2^k}{2}) + 1 \\ &= T(2^{k-1}) + 1 \end{aligned}$$

If we now set $t_k = T(2^k)$ we obtain the recurrence

$$t_k = t_{k-1} + 1$$

so that the k element of the recurrence is the value of the recurrence for 2^k (which we can do because we assumed n was always a power of 2). This recurrence in a form that we can solve because it is a nonhomoegeneous linear recurrence.

$$1 \cdot t_k - 1 \cdot t_{k-1} = 1^k \cdot k^0$$

Using the general formula with $b = 1$ and $d = 1$ we get

$$t_k = c_1 + c_2 k$$

We know need to undo our change of variables to arrive back at our original recurrence. This is a two step process:

1. Substitute $T(2^k)$ for t_k in our general solution to the recurrence:

$$T(2^k) = c_1 + c_2 k$$

2. Substitute n for 2^k and $\lg n$ for k in the equation from step 1:

$$T(n) = c_1 + c_2 \lg n$$

This is the general solution to our original recurrence before the change of variables. From this point, we proceed as normal using the initial conditions to find values for c_1 and c_2. For this recurrence, the initial condition is $T(1) = 1$. We need to determine a second initial condition using the first:

$$\begin{aligned} T(2) &= T(1) + 1 \\ &= 2 \end{aligned}$$

The 2 equations for the systems are:

$$\begin{aligned} 2 &= c_1 + c_2 \lg 2 \\ 1 &= c_1 + c_2 \lg 1 \end{aligned}$$

$$\begin{aligned} c_1 &= 1 \\ c_2 &= 1 \end{aligned}$$

The final solution is given as:

$$T(n) = 1 + \lg n$$

Example: solving the recurrence describing merge sort: ex.

$$\begin{aligned} t(n) &= 2 \cdot t(\frac{1}{2} \cdot n) + n \quad \text{with } n > 0 \text{ and } n \text{ power of two} \\ t(1) &= 1 \\ t(2) &= 3 \end{aligned}$$

1. Change of variables: $n = 2^k$

$$\begin{aligned} T(2^k) &= 2T(\frac{2^k}{2}) + 2^k \\ t_k &= 2t_{k-1} + 2^k \\ t_k - 2t_{k-1} &= 2^k k^0 \end{aligned}$$

2. Solve as an nonhomogeneous recurrence with $b = 2$ and $k = 0$:

$$\begin{aligned}(r-2)(r-2) &= 0 \\ (r-2)^2 &= 0 \\ \\ t_k &= c_1 2^k + c_2 k 2^k\end{aligned}$$

3. Change variables back to n by substituting $T(2^k)$ for t_k; and then substituting n for 2^k and $\lg n$ for k:

$$\begin{aligned}T(2^k) &= c_1 2^k + c_2 k 2^k \\ T(n) &= c_1 n + c_2 n \lg n\end{aligned}$$

4. Use initial conditions to find value for c_1 and c_2:

$$\begin{aligned}1 &= c_1(1) + c_2(\lg 1) \\ 3 &= c_1(2) + c_2(2 \lg 2)\end{aligned}$$

$$\begin{aligned}c_1 &= 1 \\ c_2 &= \frac{1}{2}\end{aligned}$$

The final solution is given as:

$$T(n) = n + \frac{n}{2} \lg n$$

From this, we can see that $T(n) \in \Theta(n \lg n)$, which thankfully, is the result we expected.

3.2 Homework

Solve the following recurrences using the above methods:

1. $T(n) = 2T(\frac{n}{3}) + \log_3 n$ for $n > 1$, n a power of 3, and $T(1) = 0$
2. $T(n) = 10T(\frac{n}{5}) + n^2$ for $n > 1$, n a power of 5, and $T(1) = 0$
3. $nT(n) = (n-1)T(n-1) + 3$ for $n > 1$ and $T(1) = 1$

Do not spend more than 2 hours working on this.

4 Summary

At this point we have three main "recipes" for solving recurrences. And to be honest, we really only know how to solve linear homogeneous recurrences with constant coefficients. The other two kinds of recurrences we solve by translating them into the kind we know how to solve.

To solve a linear homogeneous recurrence with constant coefficients, do the following:

1. Write down the characteristic function. This is done by setting $t_n = r^n$.

2. Solve for the roots of the characteristic function, which is a polynomial. Idealy, this is done by factoring the characteristic function and writing down the roots. In the worst case, you'll have to use the quadratic equation.

3. Write down the general solution. This is a function of the form

$$t_n = c_1 4^n + c_2 2^n$$

   where the constants may be different. Also, add extra terms for roots of multiplicity n using Equation 4 (on page 5).

4. Use the initial conditions to find the specific solution for that set of initial conditions by solving for the constants in the general solution. You may have fewer initial conditions than constants in the general solution. In this case, you'll have to pump the recurrence relation with the initial conditions to get some values.

To solve a linear nonhomogeneous recurrence with constant coefficients of the form

$$a_0 t_n + a_1 t_{n-1} + ... + a_k t_{n-k} = b^n p(n)$$

replace it with a characteristic function of the form

$$(a_0 r^k + a_1 r^{k-1} + ... + a_k)(r - b)^{d+1} = 0$$

in which d is the order of the polynomial $p(n)$. (And the order of a polynomial is the largest exponent in the polynomial). This characteristic function can then be solved using the steps to solve a linear homogeneous recurrence.

To solve a recurrence of the form

$$t(n) = l \cdot t(n \div b) + g(n)$$

(where l and b are integers and $g(n) \in \Theta(n^k)$), perform a change of variables by

1. Replace n with b^k (we have to assume that n is a power of b).

2. Replace b^k with t_k

3. Solve as a nonhomogeneous recurrence.

4. In the resulting general solution, change the variables back by replacing t_k with $T(b^k)$. Then replace b^k with n and k with $log_b n$.

5. This gives a new general solution (with the right variables). Use the initial values to solve for the constants in the general solution to obtain the solution.